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EUREKA

THE JOURNAL OF THE ARCHIMEDEANS

(The Cambridge University Mathematical Society: Junior Branch of the Mathematical Association)

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Editorial

In writing an editorial it is a virtue to be brief. I confine myself to three remarks.

Articles for Eureka are still urgently required—particularly those of a trivial or humorous variety. I shall be pleased to receive contributions at any time, especially during the Lent Term, so that our next issue can be produced before the end of the year.

I must thank all those who have helped with producing this issue in any way, by contributing articles or reading manuscripts and proofs. A special mention must be made of John Leech, who undertook the difficult and tedious job of preparing the diagrams for the block-makers.

My last remark is to express my thanks and appreciation of the work of our business managers who have dealt with thousands of letters in connection with the magazine, in addition to attending to all matters concerned with advertising and finance. Its continued success depends on their efforts.

. . .

The Archimedeans

This term the membership of the Archimedeans has increased considerably, partly due to the appointment of a new officer, the College Representative Secretary. The work of administration has also been distributed more widely amongst the Committee.

Evening meetings have been well attended, especially that of Professor P. M. S. Blackett on "Cosmic Ray Showers." The other evening lectures were given by Professor T. G. Semple, Professor E. C. Titchmarsh and Professor R. A. Fisher. Two tea-time meetings and a visit to the Mathematical Laboratory were also held.

This term the Music Group has been re-formed and has been enthusiastically supported at its Tuesday afternoon meetings. The Play Reading Group, unfortunately, has lapsed. The formation of a not-too-serious Bridge Group is strongly advocated in some quarters, and the Committee would be glad to hear of anyone interested in running or supporting either of these groups.

A Christmas Party took place quite successfully and other social events, details of which will be announced later, will be held in the Lent Term.

I should like to take this opportunity of thanking those who have helped to arrange the Society's activities. We should like to hear of any others willing to help in the future.

J. S. R. C.

The Fundamental Theorem of the New Geometry

By N. A. Routledge

"Cosi Fan Tutte," as Mozart sang. The basic idea of the startling theorem that I propose to prove came to me during a recent performance of this work: "Women are all the same." What I shall demonstrate is more general—all curves are the same—but readers from Newnham and Girton will be relieved to hear that the demonstration is valid only for Euclidean spaces, and we of the relativity era know only too well that our space is non-Euclidean.

I shall give a proof only in the case of well-behaved curves in a plane. When this is proved, we can then show that any two points in the plane are the same. For let X and Y be two points in the plane. Take two circles, centres X and Y respectively. These two curves are, by the theorem, the same, so they have the same centre. Therefore X is the same as Y.

Now let X and Y be two points in a general Euclidean space. Take any plane through them. They are two points in this plane. Therefore X is the same as Y. This implies that any two configurations—curve, surface, volume, or what you will—in general Euclidean space are the same.

The rapid manner in which this very powerful result can be deduced from my theorem, combined with the wealth of simplification which must be imported by it into geometry, does, I think, justify me in calling it the "fundamental" theorem of the "new" geometry.

Now for the proof.

Let P and Q be two well-behaved curves in a plane. Now P is certainly the envelope of a family of curves (its tangents).

Let $f(x, y, \alpha) = 0$ be, for different α, the equation of the curves of such a family, and let $g(x, y, \beta) = 0$ be, for different β, the equation of curves of a family having Q as its envelope.

Eliminate first y, and then x, from these equations, giving say

$$x = \phi(\alpha, \beta) \quad y = \psi(\alpha, \beta) \qquad \ldots \quad \ldots \qquad (1).$$

For fixed α, as β varies, the equations (1) make (x, y) describe the curve $f(x, y, \alpha) = 0$.

Hence P, being the envelope of $f(x, y, \alpha) = 0$, is the envelope of the curves generated from (1) for fixed α, for different values of α.

Now P is got by eliminating α from

$$f(x, y, \alpha) = 0 \text{ and } \frac{\partial f}{\partial \alpha} = 0 \qquad \ldots \qquad \ldots \qquad (2).$$

Let us express the condition $\dfrac{\partial f}{\partial \alpha} = 0$ in terms of ϕ and ψ.

Now from (1) $\qquad dx = \dfrac{\partial \phi}{\partial \alpha} d\alpha + \dfrac{\partial \phi}{\partial \beta} d\beta$

and $\qquad dy = \dfrac{\partial \psi}{\partial \alpha} d\alpha + \dfrac{\partial \psi}{\partial \beta} d\beta.$

Therefore $dx \dfrac{\partial \psi}{\partial \beta} + \dfrac{\partial \phi}{\partial \beta} \cdot \dfrac{\partial \psi}{\partial \alpha} d\alpha = dy \dfrac{\partial \phi}{\partial \beta} + \dfrac{\partial \psi}{\partial \beta} \cdot \dfrac{\partial \phi}{\partial \alpha} d\alpha.$

But $\dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy + \dfrac{\partial f}{\partial \alpha} d\alpha = 0.$

Hence the condition $\dfrac{\partial f}{\partial \alpha} = 0$ is equivalent to

$$\frac{\partial \phi}{\beta} \cdot \frac{\partial \psi}{\partial \alpha} = \frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial \psi}{\partial \beta} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3).$$

Now $f(x, y, \alpha) = 0$ is got by eliminating β from (1).

And so, from (2) and (3), P is got by eliminating α and β from

$$x = \phi(\alpha, \beta), \quad y = \psi(\alpha, \beta), \quad \frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial \psi}{\partial \beta} = \frac{\partial \phi}{\partial \beta} \cdot \frac{\partial \psi}{\partial \alpha}.$$

But this is perfectly symmetrical, i.e. Q is got from the same equations.

Therefore P and Q are the same.

(The Editor invites readers to discover the fallacy in the proof of the theorem.)

■ ■ ■

Query

PRAY tell me, little spinning top,
Why it is you always stop
Your twisting round so steadily—
A thing you do so headily—
And wobble when inert you go.
Is it due to vertigo?

SUCRA.

Noughts and Crosses

By G. E. Felton and R. H. Macmillan

The game of noughts and crosses played on a 3^2 board is the simplest of a series of positional games in which the object is to place a number of men in a straight line before one's opponent can do so. We can subdivide such games into the *static*, in which the men remain fixed after they have been placed, and the *dynamic*, in which movement is permitted after all the men are on the board.

The theory of the 3^2 static game has been completely worked out by Dudeney, who shows in *The Canterbury Puzzles* that it should always result in a draw unless the play is restricted in certain ways. He has also examined *Ovid's Game*, which is similar, except that each player has three men only, which may be moved to adjacent cells after all are placed. Another example of the dynamic type of game is *Nine Men's Morris*, whose possibilities are not yet fully explored. Extensions of the static noughts and crosses include several practicable possibilities: in a plane, it may be required to get more than three in a line on a larger but still limited board; or the board may be unlimited. The most satisfactory of these games is *Pegotty*, in which the object is to complete a line of five men on an unlimited board. The game is of Japanese origin and sometimes called *Go-bang*, *go* being the Japanese for five.

The other obvious extension is to three or more dimensions. In three dimensions a "board" with 3^3 cells might be used, but it appears that, in general, boards with an odd number of cells are not satisfactory owing to the dominating position of the central cell; the device of omitting this cell has certain disadvantages also. This disposes of 3^3 and 5^3 boards, leaving 4^3 and 6^3 as possibilities; we first met the game on the 4^3 board in Cambridge in 1940; the object is to place four men in a line and it makes an excellent game. The only game with which we have experimented in more than three dimensions is 4^4; this appears to give a game very similar to the 4^3 one, but with certain enlarged possibilities.

The object of this note is to introduce the 4^3 game and indicate some interesting lines of play. Although the game is by no means unduly complex, we have been unable to devise a certain method of winning, so that the logical outcome with perfect play is still unknown and presents an interesting problem. Although it might be feasible to construct a wire cube in which one could play with the aid of counters, it is found convenient in practice to play on a dissected cube whose four layers are arranged side by side as in Fig. 1. Any cell can be identified by its co-ordinates, which are defined as shown. There are six fundamentally different types of line of four cells and one example of each is listed below.

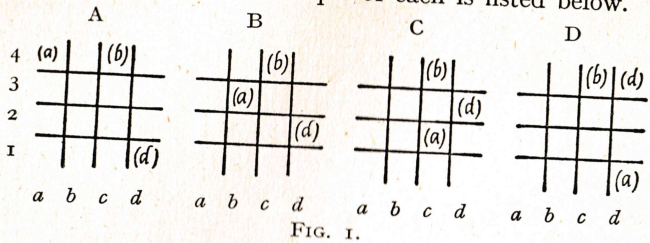


FIG. 1.

(*a*) Aa4, Bb3, Cc2, Dd1—Cube diagonal.
(*b*) Ac4, Bc4, Cc4, Dc4.
(*c*) Ac2, Bc2, Cc2, Dc2.
(*d*) Ad1, Bd2, Cd3, Dd4—Face diagonal.
(*e*) Aa3, Bb3, Cc3, Dd3.
(*f*) Ad1, Bd1, Cd1, Dd1—Edge.

TACTICS.

The aim of each player is to build up and present to his opponent such a position that a win becomes inevitable. Let us call the players A and B. If A can leave three in a line which has its fourth cell vacant, B's next move is forced for he must fill this cell. Such a position will be called a *check*. If A can contrive to leave two checks simultaneously, B must lose, for he cannot block both of them. This is the basis of the *doublet*, the simplest winning combination. A must obtain two men on each of two intersecting lines and finally fill the common cell or *node*. This configuration is very easy for B to see and also very easy to block, for a man placed on any one of three cells before A's final move will serve. It will be noted that all A's men are necessarily coplanar.

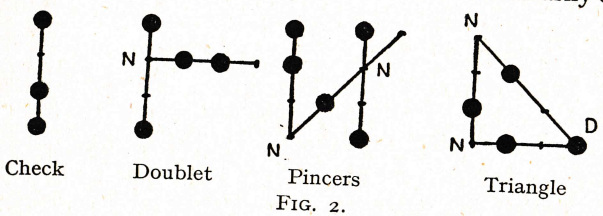
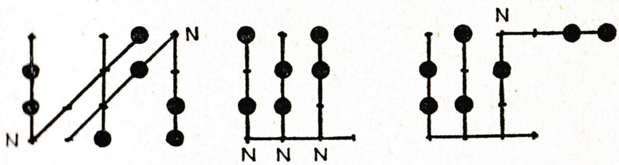


FIG. 2.

It is helpful at this stage to introduce the concept of *critical forms*, of which the check and doublet are examples. If A can build up such a configuration of his own men, B must block it (which he can often do in a variety of ways) or otherwise A at his next move can place a man at the node and so turn it into a certain win. The simpler critical forms, with our names, are shown diagrammatically in Fig. 2; the men which have been placed are marked by ●'s and the nodes by N's.

The third critical form we call the *pincers*; here A pushes B's defeat one move further back, by placing two men on each of a pair of lines intersecting a third on which he also has one man. When A places a man on either of the nodes, giving check, B cannot prevent the completion of a doublet at the next move. Six men are needed to complete this configuration, but they need not be coplanar, which makes it less likely for B to see it. The number of men required for a pincers can be reduced by arranging that one of the points shall be a *double point*, common to two of the lines. The resulting configuration, which we call the *triangle*, is necessarily plane. The idea of the pincers can, of course, be extended so that the two lines with pairs of men on them are joined by a chain of lines each with only one man on it. The nodes must be left vacant until A has completed the critical form, when a single man placed on either of the end nodes is a winning move.

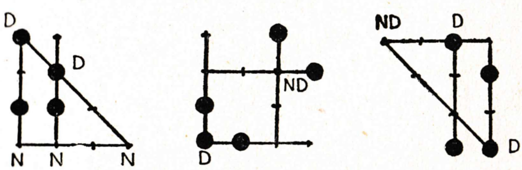
It will be noted that a single enemy man can block a configuration if it is placed on any vacant cell. It follows that constructions which are elaborate are more likely to be blocked accidentally and should not be used.



Extended Pincers (Third Rank) Triplet Extended Triplet

FIG. 3.

The pincers is a special form of the critical form we call the *triplet*, shown in Fig. 3, a configuration which can only attain its fullest freedom in four dimensions. The principle of the triplet can obviously be extended indefinitely, like the pincers. It is also possible to arrange to have one or more double points, at the



Triplet Double Doublets

Figures with Double Points

FIG. 4.

expense of becoming progressively more planar. Two such variants are shown in Fig. 4. The first variant is capable of many different elaborations and we find the remarkable fact that any four men in a plane can be turned into a winning configuration provided there is at least one *pair* amongst the four men; by the term *pair* we mean two men lying on an otherwise unoccupied line of four cells. It is not usually wise to attempt to use this configuration in any plane containing even a single enemy man.

The basic figures described above can be combined in many ways, the general principle being that the placing of the last man simultaneously completes two configurations which cannot then both be blocked at the next move. The total number of men necessary to form the configuration can be reduced by introducing double points. It may be arranged, for example, that two doublets have either two or three cells in common as in Fig. 4, the latter being merely a special case of four in a plane.

Relative Value of Cells and Lines.

The value of a cell can be assessed from the number of lines and planes in which it is contained. There are four different types of cell and their properties are given in the table below:

Type of cell.	Typical cell.	Planes.	Lines of four.
Corner	Aa1	6	7
Central	Bb2	6	7
Face	Ab2	4	4
Edge	Aa2	4	4

Similarly, the value of a line can be estimated from the number of planes in which it is contained and the number of transversals which it has. The table below gives this information for the six typical lines already enumerated. They are placed in order of merit.

Type of line.	Planes.	Transversals.
(*a*) Cube diagonal ..	3	24
(*b*) Edge	3	18
(*c*) Ac2 to Dc2 ..	3	18
(*d*) Face diagonal ..	2	18
(*e*) Aa3 to Dd3 ..	2	18
(*f*) Ac4 to Dc4 ..	2	12

Pegotty.

Apart from the fact that the game is more restricted, in that it is confined to a plane, the tactics of Pegotty are almost identical with what has been described. This depends on the fact that three men in a line constitute a check in this game also, for if A leaves this position B must place a man on the same line or A will add a fourth man and then to whichever end of the line B plays, A can put a fifth man at the other end. There is some mention of

Pegotty in the mathematical literature, but the theory of the game does not appear to have been investigated apart from the discovery of the doublet, the remark that the first player can always complete a line of four on his fifth move and the speculation that 30 to 40 moves are necessary for him to complete a line of five.

Three-dimensional Pegotty seems to offer some possibilities as one's movements are not so restricted as in 4^3 noughts and crosses; on the other hand, the board would have to be very elaborate and it might be simpler to play, for example, to get five in a line on a 6^3 board.

PROBLEMS.

In order to familiarise the reader with the notation for recording the moves and provide some recreation we append a couple of puzzles. The first is comparatively simple in that it is plane, but the second is rather more advanced.

First problem: Noughts: Aa1, Aa2, Cc4, Dd3.
Crosses: Ab3, Cb3, Da3 Dc3.

Nought is to play and win in six moves.

Second problem:
Noughts: Aa4, Ad1, Ca2, Cb4, Cc3, Cc4, Da4.
Crosses: Bb3, Bc2, Ca1, Cc2, Cd4, Db2, Dc3.

Nought is to play and win in six moves.

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Lucas, E., *Récréations mathématiques*, Paris.
Tarry, H. *et al.*, *Intermédiaire des mathématiciens*, Vol. 2, 1895, pp. 2, 194, 320.

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BACK NUMBERS

The Problems Drive

It has been customary during recent years for the Archimedeans to hold an annual Problems Drive. The competitors are divided into pairs which work together. Each pair is allowed five minutes for a problem. The winning pair is the one that produces the largest number of correct answers.

Since this issue of Eureka will be read by many hundreds of mathematicians who were not able to be present at the 1947 Drive, we have thought it worth while to include the problems for their amusement. Answers will be found on page 30.

(1) Find unequal positive integers x, y, z, such that $x^3 + y^3 = z^4$.

(2) Four explorers, A, B, C, D, can each carry sufficient food for 100 miles and can travel 25 miles a day.

They all start from base, with food, and after travelling a certain distance divide up the food. A and B return to base, where they load up with food and immediately set out again. C and D travel on, and at a certain point, divide up the food again. C goes on yet further, D returns to meet A and B. When he meets them, they share up the food they have with them, A and B return to base, load up, and set out immediately, while D goes on. Meanwhile C went so far and then turned back and continued till he met D. Then both continue back till they meet A and B again, when all four return to base.

Assuming that they would die if they ran out of food, what is the greatest distance from base that C can have reached?

(3) Determine which of these statements are true, and which false:

(i) Either (*a*) (ii) is false
or/and (*b*) both (ii) is true and (iii) is false.

(ii) Either (*a*) both (i) and (iv) are true
or/and (*b*) (iii) is false.

(iii) Either (*a*) (i) is true
or/and (*b*) both (iv) is true and (ii) is false.

(iv) Either (*a*) (ii) is true
or/and (*b*) both (i) is true and (iii) is false.

(4) Using the figures 1, 2, 3 once each, and the usual mathematical symbols, express the number 19.

(5) A. What is the greatest number that can be expressed using
(i) four 2's only
(ii) four 4's only (no symbols allowed)?

B. Give, with reasons, the next term in the series
1, 2, 4, 8, 1, 6, 3, 2, 6, . . .

(6) An engineer laid 15 wires across the Atlantic, but, unfortunately, got them mixed up. He therefore made certain connections between the ends of the wires in England, sailed across and made tests with a galvanometer in America. He then made connections at that end, returned and made similar tests in England. He was then able to work out which end in England belonged to the same wire as each end in America. How did he do it?

(The only source of E.M.F. at his disposal had to be kept with the galvanometer.)

(7) Using four 1's and mathematical symbols, express, in turn, the numbers 7, 37, 71, 99.

(8) Inside a triangle A B C a triangle A′B′C′ is constructed so that

AA′ + BB′ + CC′ is constant,

B′C′ + C′A′ + A′B′ is a minimum.

Find the position of A′B′C′.

(9) Prove that the determinant a_{ij} such that $a_{ij} = (i+j)^{i+j}$ is composite or zero.

G. C. S.

▪ ▪ ▪

AN UNSOLVED PROBLEM

GIVEN a ruler of indefinite length it is required to place on it as few marks as possible so that any integral number of inches up to n can be measured as the distance between two marks on the edge of the ruler. For example, any distance up to 13 can be measured as the distance between two of six marks arranged on the ruler at distances 0, 1, 2, 6, 10, 13 from one end.

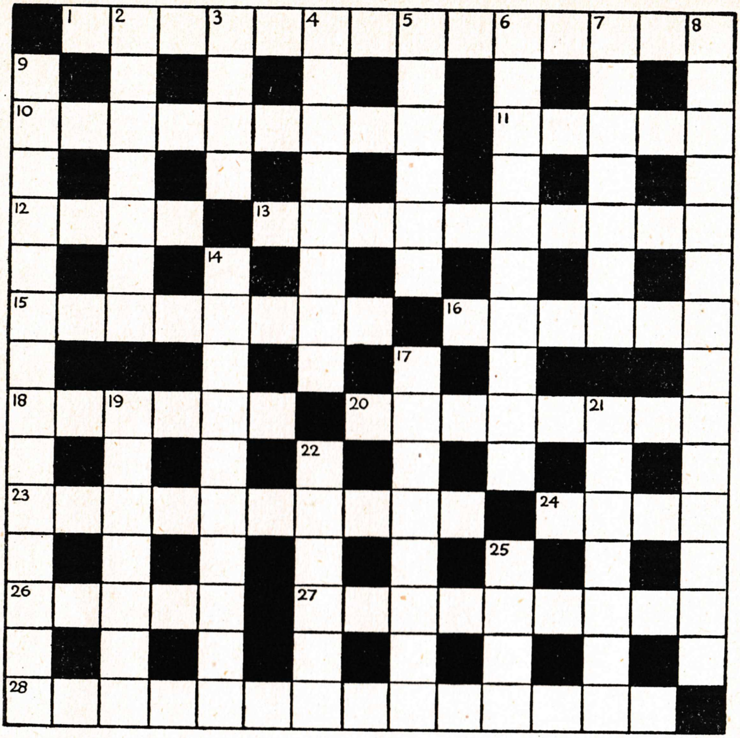
Note.—The corresponding problem for points on a circular "ruler" has been solved in some cases. Thus, if n is a prime or a power of a prime any angle which is a multiple of $2\pi/(n^2+n+1)$ can be measured as the angle at the centre between two of $(n+1)$ points arranged on a circle. This result is the best possible as there are (n^2+n+1) intervals between $n+1$ points on a circle.

▪ ▪ ▪

And does the twisted cubic know
  That howsoever far it go
It can never get away
  From those normals dropped from A?

SUCRA.

Crossword



ACROSS

1. A little question put to the classes about the Adriatic, but I'm not in on square terms (9, 5).
10. Letters for working in musical mountains (9).
11. Egypt's king or guardian (5).
12. The Romans had a word for this one (4).
13. The A.A. refine it in limited space (6, 4).
15. Jellied eels' lips are evidently oval (8).
16. One smallholder makes the market settled (6).
18. Went north on Government service in India and got the third degree three times 6).
20. Soak Ampère with tea so that he misses the note (8).
23. An angler's net is tied up with the snares (10).
24. The Heart of the Matter? (4).
26. Remove the little divot from the golf club but leave the tees, for instance (5).
27. At full length in Rome (2, 7).
28. Geometrical magnitude for keeping the rank and file together? (4, 10).

DOWN

2. Only one side in this combat (7).
3. Pascal's is Brianchon (4).
4. A letter to the skinflint helps things out (8).
5. Infuse in a small distillery (6).
6. Temporal priest has a glass among his equipment (6, 4).
7. Art part digested can be murderous (3, 4).
8. What a submarine crew feels on coming up in war-time, but it keeps things together (7, 7).
9. Three-in-one process for keeping the contents whole? (6, 8).
14. Approximately one cent in price above what caused such a disturbance (10).
17. Amicable injunction which if made by a cannibal and not complied with might cause him to omit the direction (8).
19. This one is even; which is odd (3, 4).
21. This is opposed to having a Turk in the way (7).
22. The White Rabbit, perhaps (6).
25. Lay out a French saint backwards (4).

The solution will appear in the next issue.

Ruler and Compasses

By J. Leech

Since the days of the great Greek geometers, the traditional geometrical instruments have been the ruler and compasses. So as to postulate as little as possible, Euclid used his ruler purely as a straight edge, and his compasses as an instrument with which to draw circles centred at given points and of such radii as to pass through other fixed points. In other words, his ruler can only be used to draw the straight line through two given points, and is ungraduated, while his compasses cannot be used to transfer lengths. Thus restricted, he is able by a sequence of theorems and constructions to develop his geometry to the point where he is able to inscribe the regular dodecahedron and icosahedron in a given sphere. Only did these instruments fail the Greeks when they were confronted with the three famous problems of squaring the circle, trisecting the angle, and duplicating the cube, all of which we now know to be impossible (though a few cranks still produce "solutions" to these constructions).

The more modern school of projective geometry has investigated the constructions which may be performed with the straight edge alone, giving us such constructions as finding the fourth point of a harmonic range, the sixth point of an involution, and points on, and tangents to, conics. There is even a construction for finding the ninth point through which the cubic curves through eight given general points in a plane must pass.

But the opposite problem of investigating the constructions which can be performed with the compasses alone seems to have attracted much less attention, and it is this with which the present paper deals. It will be shown that, in fact, any point which can be constructed with the ruler and compasses can be constructed with the compasses alone. A few examples of these constructions are also included.

Points which can be constructed with ruler and compasses are intersections of one of three types:

(*a*) intersections of two lines,

(*b*) intersections of line and circle,

(*c*) intersections of two circles.

To prove the desired result, it is necessary to construct intersections of types (*a*) and (*b*) by means of type (*c*) alone. The following constructions show how this may be done.

Notation used: The circle whose centre is A and whose radius is AB is denoted A(B).

A knowledge of the inversive properties of lines, points and circles is assumed.

Construction 1.—Given two points A, B, to construct points C, D, E, . . . on AB such that AB = BC = CD = DE = etc.

Draw A(B), B(A), to meet in L.

Draw L(A) to meet B(A) again in M.

Draw M(L) to meet B(A) again in C.

Then C is the first of the required points. (This is the normal practice of stepping round B(A).) To obtain D, E, etc., we repeat as necessary.

Construction 2.—Given a circle O(X) and a point A, to construct the inverse A′ of A with respect to O(X).

Case 1. $OA > \frac{1}{2}OX$.

Draw A(O) to meet O(X) in P, Q. (Fig. 1).

Draw P(O), Q(O), to meet again in A′.

Then A′ is the required inverse of A with respect to O(X). For, since the triangles OAP, OPA′ are isosceles, and have a common base angle $A\hat{O}P$, they are similar, hence $OA : OP = OP : OA'$, i.e. $OA.OA' = OP^2$. Further, by symmetry, A′ lies on OA.

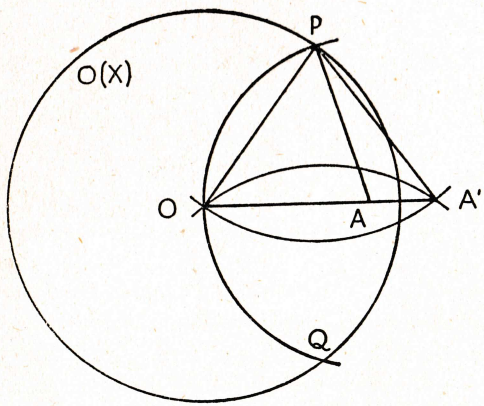


FIG. 1.

Case 2. $OA \leqslant \frac{1}{2}OX$. (A(O) does not now intersect O(X).)

Using construction 1, construct B, C, . . . K, on OA and such that $OK = k.OA > \frac{1}{2}OX$. Now construct the inverse K′ of K with respect to O(X), and construct A′ on OK′ such that $OA' = k.OK'$.

Then A′ is the required inverse, since $OA.OA' = OK.OK' = OX^2$.

Construction 3.—To construct the inverse of a circle Q(Y) with respect to a circle O(X), whose centre O is not on Q(Y).

Construct the inverse O' of O with respect to $Q(Y)$ and the inverse Q' of O' with respect to $O(X)$.

Then Q' is the centre of the required circle.

Construct the inverse Y' of Y with respect to $O(X)$.

Then $Q'(Y')$ is the required circle.

For, since O and O' are inverse with respect to $Q(Y)$, their inverses with respect to $O(X)$, which are the point at infinity and Q', are inverse with respect to the required inverse circle. Hence Q' is the centre of this circle, and as Y' is a point of this circle, its radius is $Q'Y'$. If $O(X)$ and $Q(Y)$ intersect, we may simplify this construction by choosing, instead of Y, a point of intersection of the two circles (which is its own inverse with respect to $O(X)$).

Construction 4.—To construct the inverse of the line joining two points A, B, with respect to a circle $O(X)$ whose centre O is not on AB.

Draw $A(O)$ and $B(O)$ to meet again in O', and construct the inverse O'' of O' with respect to $O(X)$.

Then $O''(O)$ is the required circle.

This is the same construction as the last with the reflection of O in the line replacing its inverse in the circle, and using the fact that the inverse of a line is a circle through the centre of inversion.

We are now able to reduce any ruler and compasses construction to a construction using the compasses only. The figure obtained by performing a construction with ruler and compasses consists only of points, lines and circles, and if the figure is inverted with respect to an arbitrary circle whose centre is on none of the lines or circles of the figure, we obtain a new figure comprising only circles whose centres and radii are known and which may be constructed with the compasses alone. The final points obtained in this inverse figure are then inverted back to give the points required in the original figure.

In particular cases, the construction may be simplified by inverting with respect to a suitable circle of the figure, or by using a construction more directly fitted to execution with the compasses, as instanced in the following constructions.

To construct the mid-point of the join of two given points A and B.

Draw $A(B)$ and $B(A)$ to meet in L and M (Fig. 2).

Draw $L(M)$ and $M(A)$ to meet in P and Q.

Draw $P(M)$ and $Q(M)$ to meet again in X.

Draw $X(L)$ and $L(A)$ to meet in R, S.

Draw $R(L)$ and $S(L)$ to meet again in O.

Then O is the mid-point of AB.

For X is the inverse of L with respect to M(A), by construction 2 and so is, by construction 4, the centre of circle ABM (which is the inverse of AB with respect to M(A)). Also, $A\hat{X}B = 2A\hat{M}B = 120° = 180° - A\hat{L}B$, so that X is on the circle ABL, and by symmetry X lies on LM. The inverse of ABL with respect to L(A) is AB, so that the inverse O of X with respect to L(A) is the required mid-point of AB.

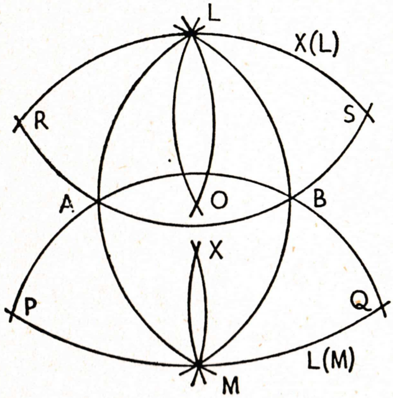


FIG. 2.

To construct the circle whose centre is A and whose radius is BC (in other words, to produce a construction equivalent to transferring the length BC to the point A, or, to justify in terms of our fictitious compasses which cannot transfer a distance, the process of transferring distances with our practical compasses).

Draw A(B) and B(A) to meet in L and M.

Draw L(C) and M(C) to meet again in D.

Then A(D) is the required circle.

For, by symmetry, LMBC and LMAD are congruent, so that AD = BC.

The problem of Archimedes, to construct the eight circles touching three given circles (*outline of constructional procedure*).

Given three circles O(X), P(Y) and Q(Z) (supposed, for convenience, of unequal radii, lettered in descending order of radii, and mutually external), to construct the two circles which respectively touch the three given circles all internally and all externally.

Construct the circles with centres O and P and with radii $OX - QZ$ and $PY - QZ$ respectively. Then the two circles through Q which touch these last two circles both internally and both externally respectively are concentric with the two required circles.

Invert this figure with respect to Q(Z). Then the two circles concentric with the two required circles invert into the direct common tangents of the inverses of the two circles last constructed. Denote these two circles J(K) and L(M), J(K) being whichever is of greater radius. Now construct the circle centre J and radius JK − LM, and the circle whose diameter is JL, and let these two circles intersect in A and B. Then LA and LB are tangents from L to J(A). Now construct the points on J(K) which lie on the lines JA and JB produced in the directions indicated, call these C and D. Complete the rectangles on LAC and LBD with E and F, then CE and DF are the required common tangents to J(K) and L(M). Now construct the centres of the circles inverse to CE and DF with respect to Q(Z), these are the centres of the two circles which touch the three given circles in the required manner. Their radii are easily found.

To construct the six circles with "mixed" contact, a similar construction is used, but using the circles centres O and P with radii $OX \pm QZ$ and $PY \pm QZ$ in pairs, and using transverse common tangents when the signs are opposite. Thus all of the eight circles are constructed. A slight modification of this construction is necessary if any two of the circles are of equal radii, while if the three circles are not mutually external, certain of the circles are imaginary. In particular cases, one or more of them may degenerate into lines. Thus we may perform the construction of all of the real circles which touch the three circles with the compasses alone. In fact, the only things you can't do with the compasses is use them as rulers, and even then, the construction of lines is unnecessary in the construction of points.

Who wants to go on using rulers?

Bibliography

L. Mascheroni: *La Geometria del Compasso* (1797) (French translation available in the University Library). This work gives an extremely large number of constructions using the compass alone, most of them very complicated, as inversion had not then been invented.

Treatments similar to the present paper are in:

J. L. Coolidge: *Treatise on the Circle and Sphere*, pp. 186–188.

H. G. Forder: *Foundations of Euclidean Geometry*, pp. 221–223. The present notation O(X) for the circle centre O and radius OX is due to Forder.

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TWO PROBLEMS ON CATS

(1) Two cats are sitting on a roof. Which is more likely to slip off first?

(2) Prove that one cat has nine tails.

Members of Parliament—A Problem in Abstract Algebra

By H. A. Thurston

The Parliament of a certain country is electing members of itself to certain posts. Each election is by a board consisting of president, vice-president and secretary. As any M.P.—there are N of them—can play two or even three parts on the same board there are altogether N^3 distinct boards, each of which makes just one election. There is no case of the same man being elected by two boards differing by only one officer. Moreover, for any five M.P.'s the board whose president is the man elected by the board whose officers are the first, second, and third men respectively, whose vice-president is the fourth man and whose secretary is the fifth man always makes the same choice as the board whose president is the first man, whose vice-president is the man elected by the board whose officers are the second, third and fourth men respectively, and whose secretary is the fifth man.

Prove (i) That this choice coincides with that of the board whose president is the first man, vice-president the second, and secretary the man elected by the board whose officers are the third, fourth and fifth men respectively.

(ii) That if A is any M.P. there is an M.P. B such that any board with A and B as president and vice-president (not necessarily respectively) will always elect the secretary.

(iii) If each man when serving as all three members of a board always elects himself, then interchange of president and secretary of any board does not affect its choice.

(iv) If, in addition, the same three members always make the same choice, no matter which holds which office on the board, then N must be even.

(v) If there is a man X who whenever he serves as two members of a board always elects the third and if we define the product of two M.P.'s A and B as the M.P. elected by the board whose officers are A, B and X respectively, then the M.P.'s form a group.

The solution appears on page 31.

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In the Greek mathematical forum
Young Euclid was present to bore 'em.
He spent most of his time
Drawing circles sublime,
And in crossing the Pons Asinorum.

A Note on Fermat's Last Theorem and the Mersenne Numbers

By C. B. HASELGROVE

THE object of this paper is to establish a connection between Fermat's Last Theorem and some numbers which are of the same type as the Mersenne Numbers but which are more general in nature. A table of these numbers, which we shall call the *Associated Mersenne Numbers*, can be found at the end of this paper. The method that we shall use is the classical method of the theory of equations which we shall apply to the theory of congruences. We shall assume that the reader is familiar with the elementary theory of congruences as given in works such as Hardy and Wright: *An Introduction to the Theory of Numbers*. Almost all the theorems of the theory of equations may be taken over into the theory of congruences by merely replacing the equality signs by congruence signs. In particular, this is true of the theorem that any symmetric function of the roots, with integral coefficients, can be expressed as a polynomial function of the coefficients with integral coefficients. The proof of this result in the theory of congruences is the same as in the theory of equations except for the replacement of all the equality signs by congruence signs.

It is well known that if p is a prime of the form $(nr + 1)$ the congruence:

$$x^n \equiv 1 \pmod{p} \qquad \dots \quad \dots \quad \dots \qquad (1)$$

has n distinct roots which are the residues which r^{th} powers may take (mod p). For by a theorem due to Fermat we have $a^{nr} \equiv a^{p-1} \equiv 1 \pmod{p}$ provided that p does not divide a. For if x is a root of the congruence (1) the congruence $a^r \equiv x$ has at most r roots. Also the congruence (1) has at most n roots. If it has fewer than n roots we arrive at a contradiction since a can take nr different values (mod p). Let the roots of the congruence (1) be $x_1, x_2, \dots x_n$. Then, as we have stated above, any polynomial symmetric function of the x_i with integral coefficients can be expressed as a polynomial function of the coefficients with integral coefficients. This function of the coefficients is the same as the corresponding symmetric function of the roots of the equation

$$x^n = 1 \qquad \dots \quad \dots \quad \dots \quad \dots \qquad (2)$$

which we shall suppose has roots $z_1, z_2 \dots z_n$ where $z_n = 1$. Thus we have in particular

$$\Pi \, (n_i + x_j - 1) \equiv \Pi \, (z_i + z_j - 1) \pmod{p} \qquad \dots \qquad (3)$$

where i and j both run from 1 to n on both sides of the equation. As the factors of the left-hand side of (3) are the possible values of

$x^r + y^r - 1 \pmod p$, the necessary and sufficient condition that it is possible to solve the congruence

$$x^r + y^r \equiv 1 \pmod p \quad \ldots \quad \ldots \quad (4)$$

is that p should divide the right-hand side of the equation (3), which is an integer which we shall denote by $\sigma(n)$. This is the necessary and sufficient condition that the congruence

$$x^r + y^r \equiv z^r \pmod p \quad \ldots \quad \ldots \quad (5)$$

can be solved with xyz not divisible by p. For if we can solve (4) we can certainly solve (5) by taking $z \equiv 1$. Also, if we can solve (5) we can solve (4) by finding a, so that $az \equiv 1 \pmod p$ and then multiplying both sides of the congruence (5) by a^r. Hence, if x, y and z are three positive integers such that:

$$x^r + y^r = z^r \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

and if p is a prime of the form $(nr + 1)$ then either p divides xyz or divides $\sigma(n)$. Thus, p divides $xyz\sigma(n)$. It now remains to determine the factors of the numbers $\sigma(n)$.

Consider the product

$$a_k(n) = \Pi\ (z_i^k + z_i - 1) \quad i = 0, 1, \ldots n - 1 \quad \ldots \quad (7).$$

Then $a_k(n)$ is an integer since the product on the right-hand side of (7) is a symmetric function of the roots of the equation (2). Further, if n is a prime we have:

$$\prod_{k=1}^{n-1} a_k(n) = \prod_{i=1}^{n} \prod_{k=1}^{n-1} (z_i^k + z_i - 1).$$

Now if $z_i \neq 1$, z_i^k runs through all the $z_j \neq 1$. If $z_i = 1$, $z_i^k + z_i - 1 = 1$ for all k. Hence the product equals

$$\prod_{i=1}^{n} \prod_{j=1}^{n} (z_i + z_j - 1)$$

since the product of those terms with $z_i = 1$ or $z_j = 1$ is 1.

Thus

$$\sigma(n) = \prod_{k=1}^{n-1} a_k(n) \quad \ldots \quad \ldots \quad \ldots \quad (8).$$

Also for composite n we see that $a_k(n)$ divides $\sigma(n)$. Thus by studying the properties of the numbers $a_k(n)$, which we shall call the *Associated Mersenne Numbers*, we can obtain information about the numbers $\sigma(n)$. Suppose that the roots of the equation

$$z^k + z - 1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

are $b_1, b_2 \ldots b_k$, where $k \geqslant 2$. Then since $\Pi\ (b - z_i) = b^n - 1$ we have

$$a_k(n) = \Pi\ (1 - b_j^n) \text{ where } j \text{ runs from 1 to } k \quad \ldots \quad \ldots \quad (10)$$

This expresses $a_k(n)$ as a symmetric function of the roots of the equation (9). We shall now state some results that can be deduced from (10); proofs will not be given here as they involve the theory of the Galois Imaginaries. For an account of this theory see ref. 1.

(I) If n divides m then $a_k(n)$ divides $a_k(m)$.

(II) If p and q are primes and if p divides $a_k(q)$ then q divides p^K-1 where K is the lowest common multiple of 1, 2, . . . k.

(III) If p is a prime then p divides $a_k(p^K-1)$, and the residues of $a_k(n)$ (mod p), as a function of n, repeat with period p^K-1.

(IV) There is a linear recurrence formula for $a_k(n)$ regarded as a function of n. For example, we have:

(i) $a_1(n) = 2^n - 1. \qquad a_1(n) = 2a_1(n-1) + 1.$

(ii) $a_2(n) = -a_2(n-1) + a_2(n-2) + 1 - (-1)^n.$

(iii) $a_3(n) = a_3(n-1) - a_3(n-2) + 3a_3(n-3) - a_3(n-4) + a_3(n-5) - a_3(n-6).$

The result (I) is a trivial consequence of the formula (10), for the quotient $a_k(m)/a_k(n)$ is clearly a symmetric polynomial function of the roots of (9) and so is an integer.

The linear recurrence formulae may easily be proved by multiplying out the product for $a_k(n)$. This expresses $a_k(n)$ as the sum of the n^{th} powers of certain quantities which may be regarded as the roots of an equation with integral coefficients. It is shown in books on algebra (e.g. ref. 2) that such an expression satisfies a linear recurrence relation with the same coefficients as the equation.

The results (II) and (III) may easily be proved by means of the Galois Imaginaries which enable us to solve the congruence $z^k + z - 1 \equiv 0$ (mod p). The relation (II) shows the analogy between the numbers $a_k(n)$ and the Mersenne Numbers which satisfy the relation (II) with $k = 1$, $K = 1$.

As the sign of the numbers $a_k(n)$ is irrelevant to the subject, we have tabulated them as if they were positive numbers. There is something to be said for modifying the definitions so that they are necessarily positive. The tables have been constructed by using the linear recurrence formulae. The relations (I), (II) and (III) form a very useful check on the accuracy of the calculations. There are several very interesting relations between the numbers $a_k(n)$ which there is no space to discuss here. For example, $a_k(n) = a_l(n)$ if $kl \equiv 1$ (mod n). It would be very interesting to study under what conditions $a_k(p)$ is prime, but the author has not the time at his disposal to carry out any of the laborious calculations necessary. It is possible that the numbers may provide a useful test for the primality of the Mersenne Numbers and other related numbers. The numbers $a_2(n)$ have already been used for this purpose by Lucas (ref. 3).

References

1. L. E. Dickson, *Linear Groups*, Chapters I–V.
2. Durell and Robson, *Advanced Algebra*, Vol. II, Chapter XI.
3. Hardy and Wright, *Theory of Numbers*, pp. 147 and 243.

Table of the Associated Mersenne Numbers

n	$k = 1$	$k = 2$	$k = 3$
0	0	0	0
1	1	1	1
2	3	1	3
3	7	4	1
4	15	5	3
5	31	11	11
6	63	16	9
7	127	29	8
8	255	45	27
9	511	76	37
10	1023	121	33
11	2047	199	67
12	4095	320	117
13	8191	521	131
14	16383	841	192
15	32767	1364	341
16	65535	2205	459
17	131071	3571	613
18	262143	5776	999
19	524287	9349	1483

Remark on the Motion of Tops in reply to Query

Dear "Sucra",

Though with great success
At first I steadily precess,
This later changes to nutation—
A thing you'll find by computation—
And now please let the matter drop.

Signed,

Yours,

A mathematic

Top.

P.S.—For further reference, Lamb,
The latter part of his Dynam.

E.D.S.A.C.

By D. J. Wheeler

Electronics has only recently invaded the field of calculating machines. It was first used in E.N.I.A.C., the electronic numerical integrator and calculator made in America during the war. Whilst this was under construction it was observed that machines having an equivalent performance could be built with fewer valves if they incorporated high-speed memories. Such machines are now being constructed in various places throughout the world, the high-speed memories usually consisting of batteries of mercury tubes or cathode-ray tubes.

E.D.S.A.C. (the electronic delay storage automatic computor) is a small machine of this type being built in the mathematical laboratory under the direction of Dr. M. V. Wilkes. We shall consider briefly its mode of action.

The machine has five parts:

(*a*) A memory unit,
(*b*) A computor or arithmetic organ,
(*c*) A control unit,
(*d*) An input unit, and
(*e*) An output unit (see ref. 1).

(*a*) Numbers, expressed in the binary scale, are stored in the form of supersonic bursts of waves travelling in mercury contained in a tube. The waves are generated at one end of the tube by a vibrating quartz crystal, and travel to the other end where they are converted into electric impulses, which are amplified and used to generate waves again. The action is that of a juggler keeping many balls in the air at the same time. Each memory tube can juggle with 576 digits, and since the memory consists of two batteries of 16 tubes, it can hold 1024 numbers, each of 17 digits. Any of these numbers can be read or replaced at will.

(*b*) The computor consists of a mercury delay line, called an *accumulator*. Numbers from the memory can be added to, or subtracted from, the number in the accumulator, or multiplied together and added to or subtracted from the number in the accumulator. In addition, the contents of the accumulator can replace any number in the memory. By the successive use of these elementary operations of arithmetic almost any calculation can be speedily carried out.

(*c*) The control "looks at" certain numbers called "orders" held in the memory, and interprets them as instructions, which it then executes. The order to be obeyed is specified by the sequence control number, which is increased by one as each operation is carried out. However, the sequence control number can also be altered directly by conditional transfer orders so that orders can be used over and over again—they don't wear out!

(*d*) Information is fed into the machine by punched paper tape which the machine reads when the orders instruct it to do so. There is room for five holes across the width of the tape and only one row can be read at a time. Consequently, only five binary digits are read at a time and so orders and numbers have to be assembled from the incoming digits before they can be used.

(*e*) The results of the calculation are printed by means of a modified teleprinter. This will print one symbol at a time corresponding to five binary digits.

To set the machine a problem, this must first be programmed, i.e. the orders necessary to effect the solution of the problem must be worked out. The problem, data and orders are then punched on the paper tape and fed into the machine, which carries out the calculation and prints the solution.

If each arithmetic step in the calculation had to be programmed separately, using the machine would be extremely tedious. There are, however, three facts which enable us to use a set of orders to perform a comparatively large number of operations:

(1) Orders in the memory may be used more than once.

(2) Orders in the memory can be adjusted by means of other orders.

(3) The control has a certain amount of judgment, for it can detect the difference between a positive and negative sign, and so choose one of two alternative courses of action at any point in the calculation.

The use of orders in this way is comparable with the usual shorthand notations used in mathematics to-day. Who would write:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = y$$

instead of $\sum_{1}^{10} x_i = y$?

Who would be able to write out the unshortened version of:

$$\sum_{j=1}^{100} a_{ij}\, b_{jk} = c_{ik}, \qquad i,\ k = 1,\ 2,\ \ldots\ 100,$$

which has one million terms?

Thus to calculate the scalar product of $x_1 \ldots x_{50}$ and $y_1 \ldots y_{50}$ we would programme as follows:

$$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n-1} x_i y_i + x_n y_n. \quad \text{Repeat for } n = 1, 2, \ldots 50.$$

All this implies that the machine is best suited for large numbers of similar calculations. Thus suitable problems for the machine to tackle are the solution of differential equations, Fourier synthesis, matrix multiplication, the calculation of tables, and so on. Many problems will need a different approach when machines such as E.D.S.A.C. are used to solve them (see ref. 2). "Intelligence" is likely to be very expensive to programme both in time and memory space, and so in many cases the machine will solve problems by methods which would be quite impracticable with human computation. On the other hand, some problems can be tackled more directly by the use of the machine. For instance, after programming complex addition, multiplication and division, complex numbers can be handled directly, and so functions of a complex variable can be calculated directly and not as the sum of real and imaginary parts.

Certain problems other than those of mathematics can be solved by these machines. It is possible for them to indulge in games of chess, and although it is doubtful if an existing machine could win the chess championship of the world, the possibility of one being built in a few years' time that could do so is quite real.

References

1. Dr. M. V. Wilkes, *Electronic Engineering*, July, 1948.
2. Professor D. R. Hartree, EUREKA, March, 1948.

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The Mathematical Association

President: SIR HAROLD SPENCER JONES, F.R.S.

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

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The *Mathematical Gazette* is the journal of the Association. It is published five times a year and deals with mathematical topics of general interest.

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SOLUTIONS TO CAT PROBLEMS

(1) The one with the smaller μ.

(2) No cat has eight tails.
One cat has one tail more than no cat.
Hence one cat has nine tails.

The Deltoid (II)

By A. M. Macbeath

In the first part of this article, published in the last issue of Eureka, a definition of the deltoid was given and some of its properties were proved. In this part we use some of these results to investigate the family of deltoids which touch three lines. To assist the reader we restate briefly the definition and the relevant theorems.

Let a point P describe the circumference of a circle S with uniform angular velocity, as observed from the centre of S. Let p be a line through P rotating in the *opposite* sense with *half* the angular velocity. The envelope of p is called a *deltoid.* P is called the *central point* of p. As in the last issue, we use P, P_1, P_2 . . . to denote different points on S; p, p_1, p_2 . . . the corresponding tangents of the deltoid. Q_{ij} denotes the intersection of p_i and p_j. In the last issue we proved:

T.1: *The circle* $P_1P_2Q_{12}$ *has the same radius as* S.

T.3: *The three circles* $P_1P_2Q_{12}$, $P_2P_3Q_{23}$, $P_3P_1Q_{31}$ *all pass through* O, *the circum-centre of the triangle* $p_1p_2p_3$.

T.4: *There is a unique deltoid touching four general lines.*

Consider the figure (as in T.3) of three lines p_1, p_2, p_3, touching a deltoid. Let O be the circumcentre of the triangle $p_1p_2p_3$. The three tetrads $OP_1P_2Q_{12}$, $OP_2P_3Q_{23}$, $OP_3P_1Q_{31}$ are concyclic lying on three circles all equal to the circle $P_1P_2P_3$ which is the incircle of the deltoid.

It is not difficult to show that, given a triangle $p_1p_2p_3$, there is an infinity of ways of building up such a diagram. (We leave the details to the reader.) If we assign P_1 arbitrarily on p_1, the rest of the figure is determined. Thus there is a simple infinity of deltoids inscribed to the triangle. These families have the following properties:

T.6: (i) *The centre of a deltoid touching the three sides of a triangle lies on the perpendicular bisector of* OH.*

(ii) *The incircle of a deltoid inscribed to a triangle has double contact with a certain fixed conic: namely that which touches the three lines and has* O, H *for foci.*

On applying T.4 we deduce a remarkable property of the complete quadrilateral.

T.7: *If, for each of the four triangles of a complete quadrilateral, we construct the perpendicular bisector of the segment joining circum-centre and orthocentre, the four lines so obtained are concurrent.*

* O, H denote the circumcentre and orthocentre respectively.

Proof of T.6 (i). The angles $P_1Q_{12}P_2$, $P_2P_3P_1$ are equal, standing on the common chord of two equal circles, and similarly for two other pairs. The triangles $P_1P_2P_3$, $p_1p_2p_3$ are therefore similar.

If we denote the orthocentre of $P_1P_2P_3$ by Q, we have angle $P_1QP_2 = \pi - P_1P_3P_2$, so Q lies on the circle $P_1P_2Q_{12}$. Applying a similar argument to the pairs P_2, P_3; P_3, P_1, we find, using T.3, that Q coincides with O.

Then, as P_1, P_2, P_3 vary, the triangle $P_1P_2P_3$ is fixed in shape and its orthocentre O is fixed. Every point which is fixed relatively to the triangle $P_1P_2P_3$ describes a locus similar to that of P_1, i.e. a line. In particular, the circumcentre C, which is the centre of the inscribed deltoid, describes a line. If P_1 is at the point where OP_1 is perpendicular to its locus p_1, the corresponding positions of P_1, P_2, P_3 are the mid-points of the sides, and C is at N, the nine-points centre of $p_1p_2p_3$. The locus of centres of the deltoids is thus the perpendicular through N to ON, i.e. the perpendicular bisector of OH.

T.6 (ii) is more difficult and we require two lemmas.

Lemma 1: *Let* O, H *be the common points of a coaxal system of circles. Let a variable circle of the system cut the line of centres at* C. *Let* T *be a point on the circumference such that* TC = k.OC, *where* k *is a fixed ratio. Then the locus of* T *is a conic with foci at* O, H.

In the proof of lemma 1 we use the following celebrated theorem of Appollonius: *if* P, Q *are two points, the locus of* X *such that* PX = k.QX (*where* k *is constant*) *is a circle with centre on the line* PQ. We call this locus the *k-circle* of P, Q. P, Q are found to be inverse points with respect to their k-circle, so that any circle through P, Q cuts the k-circle orthogonally.

For fixed C, T (Fig. 1) consider the two k-circles:

(1) The locus of X so that TX = k.OX; centre O′, say;

(2) The locus of Y so that TY = k.HY; centre H′, say.

Then since TC = k.OC = k.HC (from symmetry), C is on both these loci, and they both cut the circle TOCH orthogonally. Hence O′CH′ is the tangent at C; O′, H′ are respectively the intersections of this tangent with TO, TH.

O′C is the radius of the k-circle of T, O and hence O′C = m.TO where m depends only on k. Similarly, CH_1 = m.TH. Again, O′T = n.OT, where n depends only on k. Hence O′H′ = n.OH.

Thus $|OT \pm TH| = m^{-1}|O'C \pm CH'| = m^{-1}|O'H'| = m^{-1}n|OH| =$ constant, and the locus of T is a conic with foci at O, H.

Lemma 2: *Let* O, H *be fixed points and* l *their perpendicular bisector. Let* Γ *be a variable circle with centre* C *on* l *and radius* k.OC, *where* k *is a constant. Then the envelope of* Γ *is a conic with foci at* O, H. *Each circle* Γ *has double contact with this conic.*

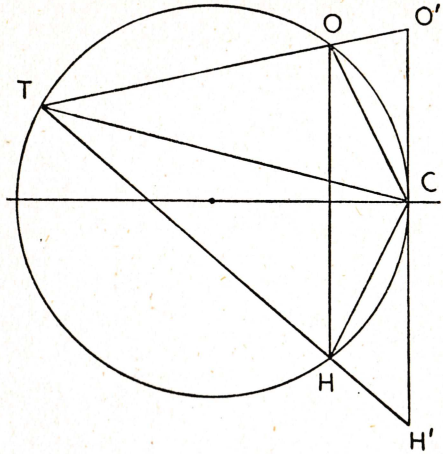

FIG. 1.

Proof: To find the envelope we consider the intersections of two neighbouring circles Γ, Γ', centres C, C′ (Fig. 2). We may suppose they meet at X, Y. Then XC′: XC = OC′: OC = YC′: YC. Thus there is a circle of Apollonius through O, X, Y passing between C, C_1 (inverse points) and having centre on l.

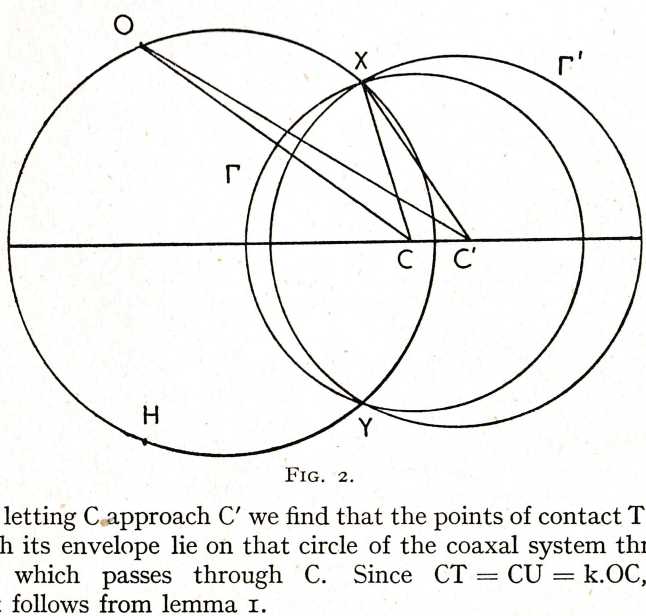

FIG. 2.

On letting C approach C′ we find that the points of contact T, U of Γ with its envelope lie on that circle of the coaxal system through O, H which passes through C. Since CT = CU = k.OC, the result follows from lemma 1.

We can now complete the proof of T.6 (ii) by showing that the set of incircles of deltoids touching p_1, p_2, p_3 satisfies the conditions of lemma 2. Regarding O, H again as the circumcentre and orthocentre of $p_1p_2p_3$, the centre C of the incircle lies on the perpendicular bisector of OH, by T.6 (i) already proved.

The triangle $P_1P_2P_3$ is fixed in shape, O is its orthocentre and C its circumcentre. Hence OC bears a fixed ratio to the circumradius, and lemma 2 shows that the envelope of circumcircles of $P_1P_2P_3$, which are incircles of the deltoids, is a conic with foci at O, H.

We leave it to the reader to show that this conic touches p_1, p_2, p_3. This can be done by proving that there is a unique deltoid of the family whose incircle touches p_1 at P_1. It is then easy to prove that, in this position, P_1 is the contact of the incircle with its envelope.

In the next issue it is hoped to include a third part of this article, giving a synthetic proof of a theorem of Morley on the Miquel points of five lines which touch a deltoid.

■ ■ ■

Book Review

The Solar System Analysed. By F. C. Attwood. (Dawson Printing Co., Ltd., Auckland, New Zealand.)

Copies of this book may be obtained from the office of the High Commissioner for New Zealand, New Zealand House, London.

This book must not be regarded as a first introduction to the theory of the solar system but rather as a set of suggestions to interested readers. Mr. Attwood has succeeded in raising many interesting topics in his small book. The approach is unorthodox—as, indeed, the title-page tells us—and many of his conclusions are original. This does not, however, condemn the book, and many of his points will repay careful study.

The book first deals with the origin of the solar system and the formation of the planets. Mr. Attwood breaks away from the idea of gravitational attraction of a neighbouring star and postulates instead an original solar nebula which, in cooling, and contracting to spherical form, leaves behind rings. These describe orbits around the central mass but in time the primitive rings contract to form planets. The formation of direct moving satellites is explained in a similar manner.

The remainder of the book contains a new theory of tidal action and interesting theories on lunar and terrestrial evolution. It is perhaps unfortunate that the book could not have been a little longer as the author has insufficient space to develop his theories. He has, however, made many suggestions which remain to be proved or disproved as our knowledge of the solar system increases.

R. J. T.

Potter's Orchard

By I. Bridges

Potter had an orchard containing 9 pear trees, so arranged that there were 9 rows each containing three trees, and no row contained more than 3 trees.

Putter gave Potter 16 young apple trees which Potter planted in his orchard in such a way that his 25 trees were arranged in 18 rows, each of five trees, and no row contained more than five trees.

What was the final arrangement of trees in Potter's orchard? A solution will appear in our next issue.

■ ■ ■

Solutions to Problems in this Issue

THE PROBLEMS DRIVE

(1) $x = 70$, $y = 105$ is one answer. See Eureka, No. 10, p. 5.

(2) 100 miles.

(3) (i) True, (ii) false, (iii) true, (iv) false.

(4) $(2/\cdot 1) - [\sqrt{3}]$ where the square brackets denote, as usual, "integral part of."

(5) A: (i) $2^{2^{22}}$ (ii) $4^{4^{4^4}}$.

B: 4. The terms are the integers of the series

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

(6) He connected the 15 wires in groups of one, two, three, four and five. In America he could pick out these groups with his galvonometer. He then selected one wire from each group, and connected them together, one wire from each of the remaining four groups, and connected them together, and so on. In England he was able to pick out these groups and so deduce which end in England belonged to each end in America.

(7) $(1 + 1 + 1)! + 1, \quad 111 \times \sqrt{(\cdot\dot{1})}$

$\cdot 1 \times (\sqrt{(1/\cdot\dot{1})})!! - 1, \quad 1/(\cdot 1 \times \cdot 1) - 1.$

(8) $A'B'C'$ is equilateral and so constructed that AA', BB' and CC' bisect its angles.

(9) If the order of the determinant, n, is greater than three, subtract the first column from the third, the second from the fourth, giving two columns of even integers. Hence the determinant is divisible by 4 or is zero. If $n = 1, 2, 3$, direct calculation (expanding by a row or column) gives the result.

Denote by (abc) the man chosen by the board whose officers are a, b and c respectively. Then as x runs through the Parliament, (xbc) represents N different men, for otherwise there would be a case of two boards differing by one member electing the same man. Hence it represents the N members just once each. Similarly, for (axb) and (abx). This means that given three of a, b, c, d, the fourth is uniquely determined by the relation $(abc) = d$.

Now we are told $((abc)de) = (a(bcd)e)$ for any $abcd$.

Then

$$(((abc)de)fg) = ((a(bcd)e)fg) = (a((bcd)ef)g) = (a(b(cde)f)g) = ((ab(cde))fg).$$

Hence $((abc)de) = (ab(cde))$, proving (i).

Given a and c there is a b for which $(abc) = c$. If x is any M.P. let $x = (cpq)$. Then $(abx) = (ab(cpq)) = ((abc)pq) = (cpq) = x$. Thus the board $(ab\,\cdot)$ always elects the secretary.

Also $(a(bab)x) = (ab(abx)) = (abx)$. Therefore $(bab) = b$. So, as above, $(bax) = x$, and so the board $(ba\,\cdot)$ also elects the secretary.

For (iii) we are told that $(xxx) = x$. So, as above, $(xxy) = y$.

Then $((xyz)(xyz)(xyz)) = (xyz) = (x(yyy)z) = (x((yzz)y(xxy))z)$
$= ((xyz)(zyx)(xyz))$. Hence $(xyz) = (zyx)$.

Under the conditions of (iv) we can associate with any M.P. p a partner (pab). Since $p = (paa) \neq (pab)$ no M.P. is his own partner. (pab)'s partner is $((pab)ab) = (pa(bab)) = (paa) = p$. Hence we have divided N into pairs, so N must be even.

For (v) the divisibility postulates are obvious.

$[ab]c = ((ab\mathrm{X})c\mathrm{X}) = (ab(\mathrm{X}c\mathrm{X})) = (abc) = ((abc)\mathrm{XX}) = (a(bc\mathrm{X})\mathrm{X})$ $= a[bc]$ where square brackets denote the product as defined. Thus the associative law holds and the M.P.'s form a group.

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Solutions to Problems in Eureka No. 10

A TENNIS PROBLEM

In any knock-out tournament the number of matches played is one less than the number of entrants. Hence there were 99 matches in all. The other information given was irrelevant.

CROSSWORD PUZZLE

Across:—1. Indeterminate. 8. Card. 9. Definition. 11. Machine. 12. Grad. 14. Lily White. 16. Etnas. 17. Xenon. 19. Tasteless. 21. Oner. 22. Confine. 24. Birational. 25. Bias. 26. Apple of his Eye.

Down:—2. Direct Line. 3. Reeve. 4. Icing Sets. 5. Alidade. 6. Edit. 7. Linear Systems. 8. Complex Number. 10. View. 13. Indefinite. 15. Interpose. 18. Non-stop. 20. Eons. 22. Chaff. 23. Area.