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EUREKA

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Editorial

THIS year has seen the retirement of the Sadleirian Professor of Pure Mathematics, PROF. L. J. MORDELL, F.R.S. The greater part of Prof. Mordell's important contributions to mathematics has lain in the field of Arithmetic. Lest our readers should suspect the Sadleirian Professor of doing "sums" we should add that we are referring to certain branches of the Theory of Numbers. In particular, he has made fundamental contributions to the theory of Diophantine equations, to the Geometry of Numbers, and to the theory of elliptic modular functions. Under his guidance the study of these subjects has born much fruit in Cambridge in recent years. We wish Prof. Mordell a long, happy, and active retirement.

We welcome as his successor MR. PHILIP HALL, F.R.S., the former Reader in Algebra. Mr. Hall has made many fundamental contributions to the Theory of Groups and, in particular, to the theory of the structure of finite groups. When he took the Mathematical Tripos there were no lectures delivered in Cambridge on Modern Algebra. It is a tribute to his influence and achievements that there are now so many lecture courses on this most important subject. We are fortunate in having so worthy an upholder of the high traditions of this Chair.

The Archimedeans

THE Society's numerous activities continue to flourish and membership remains as high as ever. In the Michaelmas Term Mr. Hubert Phillips, "the Einstein of Fleet Street," spoke on football pools and other ways of making money, Professor Whitehead mapped spheres, Professor Peierls iterated and Professor Bartlett discussed population problems. In the Lent Term we had Professor Heilbronn who puzzled and delighted us with Hieroglyphics, Sir George Thomson who delved into the Laws of Nature and Dr. C. A. B. Smith who showed us how to Square the Square with the help of slides.

In addition there was a popular and successful visit to the Cavendish, and the Annual Problems Drive, which resulted this year in a record number of scratched heads. The tea-time meetings, addressed by Messrs. Zeeman, Harris, Pirani and Rickayzen, were enlivened by the use of plasticene and rotating buckets, thus setting a precedent which we hope will be followed by others in the Faculty.

The financial obstacles to holding the Christmas Party and the Dance were overcome, both events proving a great success. Musical entertainment was provided both by the weekly meetings of the

Music Group and by a visit to a Gilbert and Sullivan performance at the Arts Theatre.

In thanking last year's committee and wishing luck to the new one, I must record how much we shall miss the loss of our hard-working Secretary to the Forces.

M. F. A.

A Contribution to the Mathematical Theory of Big Game Hunting

By H. PÉTARD

(Reprinted from the *American Mathematical Monthly*, Vol. XLV, No. 7 (1938), by kind permission of M. Pétard and with the kind co-operation of the Editor)

THIS little known mathematical discipline has not, of recent years, received in the literature the attention which, in our opinion, it deserves. In the present paper we present some algorithms which, it is hoped, may be of interest to other workers in the field. Neglecting the more obviously trivial methods, we shall confine our attention to those which involve significant applications of ideas familiar to mathematicians and physicists.

The present time is particularly fitting for the preparation of an account of the subject, since recent advances both in pure mathematics and in theoretical physics have made available powerful tools whose very existence was unsuspected by earlier investigators. At the same time, some of the more elegant classical methods acquire new significance in the light of modern discoveries. Like many other branches of knowledge to which mathematical techniques have been applied in recent years, the Mathematical Theory of Big Game Hunting has a singularly happy unifying effect on the most diverse branches of the exact sciences.

For the sake of simplicity of statement, we shall confine our attention to Lions (*Felis leo*) whose habitat is the Sahara Desert. The methods which we shall enumerate will easily be seen to be applicable, with obvious formal modifications, to other carnivores and to other portions of the globe. The paper is divided into three parts, which draw their material respectively from mathematics, theoretical physics, and experimental physics.

The author desires to acknowledge his indebtedness to the Trivial Club of St. John's College, Cambridge, England; to the M.I.T. chapter of the Society for Useless Research; to the F. o. P., of Princeton University, and to numerous individual contributors, known and unknown, conscious and unconscious.

1. Mathematical Methods

1. *The Hilbert, or Axiomatic, Method.* We place a locked cage at a given point of the desert. We then introduce the following logical system.

Axiom I.	The class of lions in the Sahara Desert is non-void.
Axiom II.	If there is a lion in the Sahara Desert, there is a lion in the cage.
Rule of Procedure.	If p is a theorem, and "p implies q" is a theorem, then q is a theorem.
Theorem I.	There is a lion in the cage.

2. *The Method of Inversive Geometry.* We place a *spherical* cage in the desert, enter it, and lock it. We perform an inversion with respect to the cage. The lion is then in the interior of the cage, and we are outside.

3. *The Method of Projective Geometry.* Without loss of generality, we may regard the Sahara Desert as a plane. Project the plane into a line, and then project the line into an interior point of the cage. The lion is projected into the same point.

4. *The Bolzano-Weierstrass Method.* Bisect the desert by a line running N–S. The lion is either in the E portion or in the W portion; let us suppose him to be in the W portion. Bisect this portion by a line running E–W. The lion is either in the N portion or in the S portion; let us suppose him to be in the N portion. We continue this process indefinitely, constructing a sufficiently strong fence about the chosen portion at each step. The diameter of the chosen portions approaches zero, so that the lion is ultimately surrounded by a fence of arbitrarily small perimeter.

5. *The "Mengentheoretisch" Method.* We observe that the desert is a separable space. It therefore contains an enumerable dense set of points, from which can be extracted a sequence having the lion as limit. We then approach the lion stealthily along this sequence, bearing with us suitable equipment.

6. *The Peano Method.* Construct, by standard methods, a continuous curve passing through every point of the desert. It has been remarked* that it is possible to traverse such a curve in an arbitrarily short time. Armed with a spear, we traverse the curve in a time shorter than that in which a lion can move his own length.

7. *A Topological Method.* We observe that a lion has at least the connectivity of the torus. We transport the desert into four-space. It is then possible† to carry out such a deformation that

* By Hilbert. See E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, 1927, Vol. I, pp. 456–457.

† H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, 1934, pp. 2–3.

the lion can be returned to three-space in a knotted condition. He is then helpless.

8. *The Cauchy, or Functiontheoretical Method.* We consider an analytic lion-valued function $f(z)$. Let ζ be the cage. Consider the integral

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-\zeta}\,dz$$

where C is the boundary of the desert; its value is $f(\zeta)$, i.e. a lion in the cage.*

9. *The Wiener Tauberian Method.* We procure a tame lion, L_0, of class $L(-\infty, \infty)$, whose Fourier transform nowhere vanishes, and release it in the desert. L_0 then converges to our cage. By Wiener's General Tauberian Theorem,† any other lion, L (say), will then converge to the same cage. Alternatively, we can approximate arbitrarily closely to L by translating L_0 about the desert.‡

2. Methods from Theoretical Physics

10. *The Dirac Method.* We observe that wild lions are, *ipso facto*, not observable in the Sahara Desert. Consequently, if there are any lions in the Sahara, they are tame. The capture of a tame lion may be left as an exercise for the reader.

11. *The Schrödinger Method.* At any given moment there is a positive probability that there is a lion in the cage. Sit down and wait.

12. *The Method of Nuclear Physics.* Place a tame lion in the cage, and apply a Majorana exchange operator§ between it and a wild lion.

As a variant, let us suppose, to fix ideas, that we require a male lion. We place a tame lioness in the cage, and apply a Heisenberg exchange operator|| which exchanges the spins.

13. *A Relativistic Method.* We distribute about the desert lion bait containing large portions of the Companion of Sirius. When enough bait has been taken, we project a beam of light across the desert. This will bend right round the lion, who will then become so dizzy that he can be approached with impunity.

* N.B. By Picard's Theorem (W. F. Osgood, *Lehrbuch der Funktionentheorie*, Vol. I, 1928, p. 748) we can catch every lion with at most one exception.

† N. Wiener, *The Fourier Integral and Certain of its Applications*, 1933, pp. 73–74.

‡ N. Wiener, *loc. cit.*, p. 89.

§ See, for example, H. A. Bethe and R. F. Bacher, *Reviews of Modern Physics*, Vol. VIII, 1936, pp. 82–229; especially pp. 106–107.

|| *Ibid.*

3. Methods from Experimental Physics

14. *The Thermodynamical Method.* We construct a semi-permeable membrane, permeable to everything except lions, and sweep it across the desert.

15. *The Atom-splitting Method.* We irradiate the desert with slow neutrons. The lion becomes radioactive, and a process of disintegration sets in. When the decay has proceeded sufficiently far, he will become incapable of showing fight.

16. *The Magneto-Optical Method.* We plant a large lenticular bed of catnip (*Nepeta cataria*), whose axis lies along the direction of the horizontal component of the earth's magnetic field, and place a cage at one of its foci. We distribute over the desert large quantities of magnetised spinach (*Spinacia oleracea*), which, as is well known, has a high ferric content. The spinach is eaten by the herbivorous denizens of the desert, which are in turn eaten by lions. The lions are then oriented parallel to the earth's magnetic field, and the resulting beam of lions is focussed by the catnip upon the cage.

4. Recent Research

Recent research has yielded the following new methods:

9*a*. *The Eratosthenian Method.* We enumerate the objects in the desert, and then examine them one by one, discarding all those which are not lions. By a refinement of this method, we can ensure that only prime lions are captured.

9*b*. *The Schwartz Method.* We can catch all the lions. However, those that we catch may not be real lions, but only distribution lions, and a competent leonologist should be on hand to discriminate.

Morley's Trisector Theorem

A well-known theorem due to F. Morley states that the three points in which pairs of internal trisectors of a triangle nearest a given side meet, are the vertices of an equilateral triangle.

There are many proofs of this theorem. None of these, however, appears to be quite as simple as the following, which is a good example of the value of "working backwards from the answer."

We shall prove that given any triangle with angles 3α, 3β, 3γ we can construct the required configuration *starting with the equilateral triangle.* (Clearly it is only the angles and not the size that are important.)

Let PQR be an equilateral triangle. Denote the angle PQR by pQr, etc. Choose L outside the triangle PQR such that

$$lQr = lRq = \beta + \gamma$$

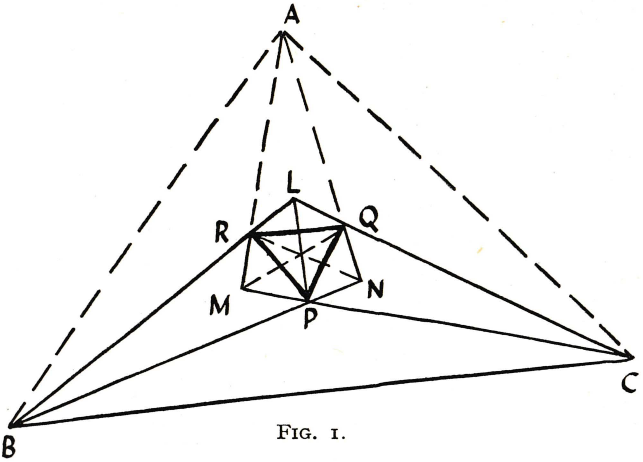

FIG. 1.

Clearly LP bisects qLr. M and N are similarly defined. Let LQ meet MP in C. Then

$$qCp = lQp + mPq - \pi = \alpha + \beta + 2\gamma - \pi/3 = \gamma$$

since $$\alpha + \beta + \gamma = \pi/3$$

Choose B on LR so that CP bisects qCb

P is the in-centre of triangle LCB and so

$$pBr = pBc = \tfrac{1}{2}(\pi - rLq - lCb) = \beta$$

giving $$bPr + rPn = \pi$$

Hence B is the intersection of LR and NP. Similarly if A is the intersection of MR and NQ we have

$$qCa = \gamma;\ qAr = qAc = \alpha$$

also $$rBa = \beta;\ rAb = \alpha$$

and so ABC is the required triangle and the theorem is proved.

R. P.

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Svoyi Kosiri is an Easy Game

IN the last issue of *Eureka* mention was made of Professor Besicovitch's Russian card game. It was said to rival chess in subtlety and to be remarkable for the simplicity of its rules. The statement about the rules was rash and has persuaded the Editor that the game is simple enough to be described in a few short sharp sentences.

The game is called "Svoyi Kosiri," or, in English, "One's Own Trumps." It can be played by two, three or four players, with the hands either concealed or exposed. The most interesting version, and the one which we shall describe, is played by two players (with their hands exposed). It originated in Russia and is most popular in Byelo-Russia. Professor Besicovitch says that it is played all over the country, but that he has rarely seen it played seriously in the towns. Professor Besicovitch is the undisputed champion of the Western Hemisphere (which for the purposes of this article is Trinity College, Cambridge), but in Russia he says he was no better than a good club player.

The game is played with ordinary playing cards, and in the Trinity College version all the cards below six are removed from the pack before dealing. The suits are dealt in a "complementary" manner; that is to say, A.'s holding in spades is identical with B.'s holding in hearts and conversely. Similarly clubs and diamonds are complementary. Thus, apart from distinctions of suit, the initial holdings are identical, and, setting aside the question of whether or not the opening lead is an advantage, the game is entirely one of skill. The best way of dealing is to deal out the 36 cards quite normally and then adjust the black suits so that they are complementary to the red ones. If the deal is such that the opening lead is a distinct advantage, the cards are adjusted by mutual agreement. An example of an initial position will be found near the end of the article.

The object of the game is to get rid of all one's cards. Between the two players there is a sort of No-Man's Land which we shall call the "discard pile." If the players both "go out" on consecutive moves, the game is a draw, but this is very rare.

In accordance with Bridge usage, we shall call the players East and West and in the sequel, when only one hand is given, it will always be West's.

As the reader will have guessed from the translation of the title, each player has his own trumps! If West's trumps are spades, then East's are the complementary suit, i.e. hearts.

The game is started by one player leading a card on to the discard pile, which at this stage consists of this one card only. The pile is always started with one card, and thereafter the cards are added in pairs. When there are cards on the pile, the player whose turn it is has a choice of two plays:

(i) He may pick up the pile. In this case his opponent starts a fresh discard pile by leading out one card.

(ii) He may "cover" the top card of the pile and then add a second card on top of the pile. His turn now ends and his opponent is faced with the same choice of play.

In (ii) the term "cover" needs explaining. A card can be covered only by a higher card in the same suit or by one of the player's own trumps. On the other hand the second card of the pair played on to the discard pile may be any card whatsoever. Obviously a trump can only be covered by a higher trump; in fact this is one of the ways of inducing one's opponent to pick up the discard pile. To play the ace of the opponent's trumps forces him to pick up the pile, and any high card in his trump suit usually has the same effect. It should be especially noted that a card can be covered with a trump even if it would be possible to follow suit! East can use his trumps to trump a lead of West's trumps just as if a plain suit had been led.

The reader should obtain a clearer idea of the game if he plays through the following easy wins. West's cards only are usually recorded, and his trumps will always be spades.

(i) With the hand below, West, with the lead, wins as follows:

S	AKQ
H	—
D	76
C	76

W leads 6D | E plays 8D and 8C | W QS 6C | 9C 9D | KS 7C | 10C 10D | AS 7D. That is, after leading 6D, W can pair off his own boss trumps with the rest of his cards and E cannot prevent W from playing off such a pair each time it is his turn to play. Obviously W, with the lead, also wins if he has n top trumps and $(n + 1)$ other cards. Similarly, with E to lead, W wins with n top trumps and n other cards. This is also true even if there is a discard pile on the table. Thus suppose there are several cards in the discard pile and it is E's turn to play, with W holding the hand below. Then E must either

S	AKQJ106
H	8
D	987
C	108

(*a*) pick up the discard pile, or (*b*) play, as his second card, 7S; for W can trump any other card with 6S and play, say, 7D; he has now 5 top trumps and 5 outside cards and has an outright win. This does not mean that E wins by doing (*a*) or (*b*), but that he has a sure loss if he does anything else. Actually he also loses if he does (*a*), as with only 6 cards outside, W's trumps AKQJ10 6 are really equivalent to AKQJ10 9, as the reader may easily verify.

(ii) West, with the lead, wins as follows:

S AJ10
H —
D 6
C 6

W 6D | 7D QS* | W picks up. E 7C | 10S 6C | 8C 8D† | JS 6D | 9D KS | AS(!) 7D | and now W has only QS left and cannot be prevented from going out next time as both KS and AS are in the discard pile!

In both the above examples, W's trumps are strong enough, and long enough compared with the number of other cards in his hand, for him to have an easy win even though the other cards in his hand are so small. (ii), for instance, is quite a clear win for W, and E should have worked this out and should not have left W in this position, but should have tried some other play, however desperate. But when trumps are weaker but outside cards stronger, then the game is most intriguing and it is usually not possible to demonstrate a win for either side. There is no longer, so to speak, a mate in five, and the players must fall back upon their ability to assess very finely the relative strengths and weaknesses of their hands.

Several general rules may be put forward tentatively.

S A10
H KQ6
D AK
C AKQ6

(i) Do not allow your trumps to be shortened too much. Strength in side suits is not often adequate compensation. For instance, with W's hand as given, E, with the lead, will probably win.

(ii) Try to keep your minor suits equal in length, and do not lose your top honours in one of them. Weakness in a minor suit means that you are forced to shorten your trumps. In (i) above E would lead 6D, and soon either W would be forced to pick up a lot of cards or he would give up his 10S.

S KQJ10 9
H —
D 76
C —

(iii) If you are missing some of your top trumps, especially the ace, beware of overestimating your position and attempting to go out too soon. For instance, in the position given, W loses even with the lead!

These general rules can be summed up by saying that one should strive for moderate strength in all the four suits. Weakness in the minor suits needs great strength in trumps to achieve a win, while, if very weak in trumps, it is often impossible to win no matter how strong the minor suits.

Now for a few brief remarks about the opening. This is a bewildering subject and it is very difficult to demonstrate the

* For the same reason as in (i) (*b*) above.

† If E plays KS here, W picks up and has 5 top trumps and 5 other cards, winning easily as in (i).

superiority of one line of play over another. Suppose the cards are dealt as below, with W to lead:—

West	S	AJ1086	East	S	KQ97
	H	KQ97		H	AJ1086
	D	AQ98		D	KJ1076
	C	KJ1076		C	AQ98

A spade or heart lead is ruled out as E would pick it up, extremely pleased with such generosity. Similarly for AD, KC, and QD. The lead of 10C is more tricky. East would be reluctant to trump with 6H or to cover with QC, and would probably pick up, though by no means sure it was good for him. Let us suppose, however, that W leads 6C. E has several replies.

(*a*) 8C and 6D. W probably plays 8D and 9H. If E picks up, W has rid himself of 6C and 8D at the expense of 9H, and is quite pleased. So E probably has to cover with 10H and must beware of his trumps becoming too weak.

(*b*) 8C and 9S. W picks up. E has got rid of 8C at the cost of 9S, which is not too good.

(*c*) 8C and 7S. W probably plays 8S and 7C. E must now take care that his clubs do not become too weak.

(*d*) 8C and 10D. Probably the best for E. W cannot very well pick up, as he has then allowed E to get rid of 8C and 10D. W must therefore cover with QD or trump with 6S.

And so the game proceeds, but it is obviously impossible to analyse it much further.

We end this article with a few disconnected remarks and speculations. First, when the reader has developed some understanding of the game, he may like to try to win, with the lead, against his opponent's holding of the four aces only! This position was introduced by Professor Besicovitch, though it would not arise in practice. Another interesting position, this time quite easily possible in practice, is AS, KH, AD, KC, which is also thought to be a loss, even with the lead. (The player holding this hand has spades as trumps.)

It is interesting to speculate if the lead is always an advantage. With certain deals it is an obvious advantage, but in general it is much more doubtful than, say, in chess. In Trinity we have played several games with the same initial position in order to clarify this question. The position we used was:

AQ1086	KJ97
KJ97	AQ1086
AQ1086	KJ97
KJ97	AQ1086

The person with the lead did not appear to have any advantage.

There is a strongly held view that to get rid of a six from your hand is worth the sacrifice of quite a valuable trump card, because later in the game it becomes very difficult to get rid of a six without overwhelming trumps. This theory was until recently very popular, so much so that the plain suit six was an automatic opening lead. The idea behind this is that a large trump will be sacrificed on the next round and the six will have been disposed of.

We may speculate also on the possibility of Svoyi Kosiri becoming as popular as chess. Amongst card games for two players it is surely chess's only rival in skill and absence of luck. It even has the advantage over chess in that the game very rarely lasts over an hour and in that there is almost always a definite result, a drawn game being most unusual.

We finish with a Svoyi Kosiri problem:

With W's trumps spades and W to lead, does he win holding JS, 10H?

R. S.
J. M. B.

In Praise of Mathematics

Although one may, at times, be bored
By Plato, drugs, or frequencies;
Convergence never wearies,
Especially of series:
Delights untold in primes are stored,
Great joys lie hid in sequences;
No Helen could surpass
The grace of Weierstrass.

It is true, beyond doubt: mathematical learning
Should always be sought by a man of discerning.
Its pleasures grow fast, like its own exponential;
The rain of its blessings is almost torrential.

T. G. P.

Geometry
Gives scope for expert commentary
By people whose brains contain proclivities
For involutions and cyclic projectivities.

R.I.M.

Quantum Field Theory

What are the natures of the elementary particles, the protons and electrons and other bricks with which our universe seems to be built? How is it that they have some wave-like properties, such as diffraction, and some particle-like properties, so that we can ascribe a mass to them? These are questions of great importance to physics and these are also questions that only find their answer in terms of the quantum theory of fields. But before we can go on to talk about that subject properly, we must understand some of the simple basic ideas of quantum mechanics itself.

Quantum mechanics may be obtained from the ordinary classical mechanics, as taught for Part I and Part II, by a well defined procedure. We take the dynamical variables (position of the particles, their momenta and energy, etc.) and, following certain rules, turn them into operators. In the classical theory these quantities, being just numbers, commute, but in quantum theory the corresponding operators do not in general commute. The physical meaning of this, is that things that in the classical theory can be simultaneously measured with arbitrary accuracy can no longer be thus measured in quantum theory. This leads to the famous "Uncertainty Principle" of Heisenberg. For example, in classical theory we can both know where a particle is and what its momentum is. In quantum theory, if we know exactly where it is it is equally likely to have any momentum; if we know its momentum exactly it is equally likely to be anywhere. This replacement of exact knowledge by the probability of obtaining certain results is characteristic of quantum theory.

We may think of these operators as being represented by matrices (actually matrices of infinite order). They can then operate on vectors and these vectors represent the possible states of the system being considered. Now vectors can be linearly combined to give other vectors and this leads to the idea of a state which is linearly dependent on other states. Since there is a state in which there is a particle at X and there is another state with a similar particle at X′, so there are states linearly dependent on these in which the particle is partly at X and partly at X′. This may seem a strange idea but what it means is this: if we try and measure where the particle is, sometimes we shall find it at X, sometimes at X′, and if we make a large number of measurements on exactly similar states our results will follow a certain probability law.

Of especial importance are the eigenstates and eigenvalues of a particular operator. All we need to know about them here is that the eigenvalues give the possible results which can be obtained by measuring the corresponding dynamical variable. This means that if an operator Z has eigenvalues 0 and 1 only, any attempt to measure Z must yield the answer either 0 or 1.

If we multiply an operator both behind and before by a vector we obtain a number, in general complex. This is termed a *matrix element*. We are usually concerned with finding these matrix elements as we can calculate probabilities in terms of the squares of their moduli.

Quantum mechanics, as briefly described above, is applied to systems having only a finite number of degrees of freedom. The quantum theory of fields is concerned with the application of these ideas to systems having an infinite number of degrees of freedom. An example of such a system is provided by the electromagnetic field. The interesting problem is to find the eigenvalues of the momentum and energy operators of the field. It turns out that these are the same as those which would be obtained if we thought of the field as being made up of particles having definite momenta and energy. These particles are, of course, photons. Thus the quantum theory of fields leads us quite naturally to the dual "particle-wave" picture of light that had been such a puzzle to physicists for so long a time.

These ideas can be applied to other fields with interesting results. Some fields rather like the electromagnetic field give us a description of the particles called mesons which are produced by cosmic rays and high energy nuclear reactions. Ordinary quantum theory introduces wave fields to describe the behaviour of a single electron. These fields can also be quantised by a similar procedure (certain changes are necessary which need not concern us here) and the resulting quantum field describes systems with many electrons present.

All this work is mathematically unexceptionable but physically useless. It is mathematically unexceptionable because no infinities arise and all quantities are well defined: it is physically useless because all the fields considered are not allowed to interact with each other. As soon as we allow the fields to interact we obtain infinite quantities appearing everywhere.

Much work has been done on this problem in recent years and ways of obtaining sensible results from these infinite quantities have been found which agree with experiment with almost embarrassing accuracy. The most picturesque work has been that due Feynman who has developed a brilliant technique for writing down matrix elements from space-time pictures called graphs. Such a picture is shown in Fig. 1. At A the electron (represented by a solid line) emits a photon (represented by a dotted line) which is reabsorbed at B. But in between A and B the electron is going backwards in time. What does this mean? Feynman's answer is

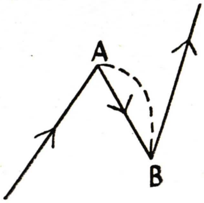

FIG. 1.

simple. An electron going backward in time is the same thing as a *positron* going forward in time! Thus Fig. 1 represents an electron-positron pair annihilating at A and creating a photon which itself creates an electron-positron pair at B.

Despite the great advances of recent years the quantum theory of fields has not reached its final form as yet and there remain many difficult problems to be solved.

BODMAS.

Problems Drive

1. Show that in any triangle there can be inscribed, by ruler and compass construction, an equilateral triangle.

2. The occupiers of six successive houses in a row all have different papers delivered by the same newsboy. On 1st April the newsboy shuffled the papers thoroughly and then delivered them in their new order, one to each house as he went along the row in his normal way. What is the chance that no person's paper is further away than at his next door neighbour's house?

   What would be the chance if there were seven houses?

3. The function $f(rstu)$ is defined to satisfy:

$$f(rstu) = f(turs) = -f(srtu)$$
$$f(rstu) + f(rsuv) = f(rstv)$$

Prove: $$f(rstu) + f(trsu) + f(stru) = 0$$

4. Four boys, Richard, Charles, John and Bryan live in that order round a circular terrace. When they visit each other they go straight across. Charles finds that he takes 40 sec. to walk from his home to Richard's and 35 sec. to John's. Bryan runs from his house to John's in 20 sec. and to Richard's in 15 sec. Which takes longer, Bryan to run from Richard's to John's or Charles to walk from his house to Bryan's? Prove your answer.

   Estimate limits for the ratio of their speeds.

5. Give the next term of each of the sequences:—

(a)		(b)	
	1		1
	24		24
	281		281
	2824		2824
	28281		28282
	......		......

6. A gambling game is played as follows:—

A player throws the same dice twice, and the scores are added up.

If the score adds up to an *odd* number, the player wins.

If it adds up to an *even* number, he loses.

In the unlikely event of a house allowing a dishonest player to provide his own dice, how would he bias the dice, in order to give himself an improved chance of winning?

7. A plane network N consists of straight lines, exactly three of which meet at every vertex.

It is *not* always possible to colour the lines of the network with three colours, so that no two lines of the same colour meet at a vertex.

Give an example to illustrate this.

E.g., the network shown in Fig. 1 is coloured in the required manner.

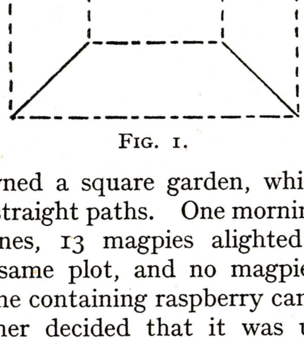

FIG. 1.

8. A gardener owned a square garden, which was divided into plots by five straight paths. One morning, as he was pruning raspberry canes, 13 magpies alighted. No two magpies were in the same plot, and no magpie alighted in a plot adjacent to one containing raspberry canes.

The gardener decided that it was unlucky to work in company with 13 magpies; he therefore dropped his secateurs, and left immediately, leaping over the fence.

What is the greatest number of paths it may have been necessary for him to cross in order to reach the fence?

Prove your answer.

(Note: (i) We say plots are adjacent if and only if they have a side in common.

(ii) There was no other path in the garden, beyond the five mentioned.)

9. In 1611, the gold coins current in this kingdom were:—

30/–, 20/–, 15/–, 10/–, 5/–, 4/–, 2/6.

In that year, King James I caused it to be proclaimed that the value of all gold coins should be raised by 10 per cent. (the value of each coin being calculated to the nearest farthing, if necessary).

This edict remained in force until 1618; in, say, 1615, what was the least integral number of pounds which could be paid exactly, in gold, without change being required? (N.B. the correct answer is less than £10.)

Further, how many coins were necessary, in order to make a payment of exactly £100?

(In both cases, give the coins used.)

10. Figure 2 represents a balance, with two arms, jointed at B. It has three scale pans provided, as shown, and you are given that

$$AX = XB, \ YC = 2BY.$$

(You are allowed to place weights in the same pan as the substance to be measured.)

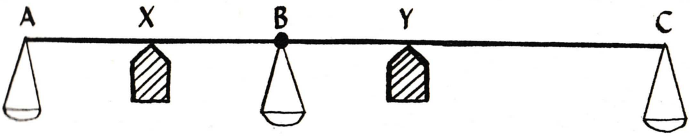

FIG. 2.

(*a*) How many weights are necessary for it to be possible (at a single weighing) to measure any integral number of ounces up to 42?

(*b*) Up to about how many stone is it possible to weigh, accurate to an ounce, given six weights?

(*Solutions on page 21.*)

Why are Series Musical?

ASKS BLANCHE DESCARTES

MOST mathematicians know the theory of the game of Nim, described in books on mathematical recreations. But few seem to be aware of Dr. P. M. Grundy's remarkable generalisation, published in *Eureka* **2**, 6-8 (1939). Consider a game Γ in which 2 players move alternately, and the last player wins (moving to a "terminal position"). Define inductively a function G(P) of the position P [Ω(P) in Grundy's Paper] as follows:—

(*a*) if P is terminal, $G(P) = 0$,

(*b*) if there are permitted moves from P to Q, from P to R, from P to S, and so on, then G(P) is the least non-negative integer different from all of G(Q), G(R), G(S), etc.

It follows that if $0 \leqslant r < G(P)$ there is a move from P to some R with $G(R) = r$, but no move to any position U with $G(U) = G(P)$. If positions P with $G(P) = 0$ are called "safe," the winning strategy is to move always to a safe position: either this is terminal, and wins immediately, or the opponent moves to an unsafe position and the cycle repeats.

Now imagine the players engaging in a "simultaneous display" of k games $\Gamma_1, \Gamma_2, \ldots \Gamma_k$ of this sort, the rule being that each player in turn makes a move in one and only one game, or if he cannot move in any game he loses. Let $P_1, P_2, \ldots P_k$ be the positions in the respective games $\Gamma_1, \Gamma_2, \ldots \Gamma_k$. Then Grundy's Theorem states that—

(i) this combined position is safe if and only if k heaps of $G(P_1)$, $G(P_2), \ldots G(P_k)$ counters respectively form a safe combination in Nim,

(ii) more generally, the G function of the combined position is the "nim-sum" of the separate $G(P_s)$ (i.e. obtained by writing the $G(P_s)$ in the scale of 2 and adding columns mod 2).

For no player can gain any advantage by moving so as to increase any $G(P_s)$, as the opponent can restore the *status quo*. And if only decreases in $G(P_s)$ are considered, the game is identical with Nim, thus proving assertion (i). Therefore $G(P) = g$ if and only if the combined position (P, P′) is safe, where $G(P') = g$. From that (ii) follows fairly readily.

It follows that we can analyse any such combined game completely, provided that we can find the $G(P_s)$ for the component positions. Nim is an example; a heap H_x of x counters constitutes a component position, since each player in turn alters one heap only, and $G(H_x) = x$. Many variants of Nim are similarly analysed.

Less trivial is Grundy's game, in which any one heap is divided into two unequal (non-empty) parts. Thus heaps of 1, 2, are terminal, with $G = 0$, a heap of 3 can only be divided into $2 + 1$, which is terminal, so $G(H_3) = 1$. Generally $G(H_x)$ in Grundy's game is the least integer $\geqslant 0$ different from all nim-sums of $G(H_y)$ and $G(H_{x-y})$, $0 < y < \frac{1}{2}x$. The series goes

$x =$	0	1	2	3	4	5	6	7	8	9	10	11	12
$G(H_x) =$	0	0	0	1	0	2	1	0	2	1	0	2	1

continuing with 3, 2, 1, 3, 2, 4, 3, 0, 4, 3, 0, 4, 3, 0, 4, 1, 2, 3, 1, 2, 4, 1, 2, 4, 1, 2, . . . This curious "somewhat periodic series" seems to be trying to have period 3, but with jumps continually occurring. Mr. Richard K. Guy confirmed this up to $x = 300$. He suggested that it might be played on a piano, taking 0 to be middle C, $1 = D$, $2 = E$, etc. The inner meaning then became evident:

FIG. 1.

Guy also worked with rows R_x of x counters, in which certain sets of consecutive counters could be extracted (thus possibly leaving two shorter rows, one each side of the extracted set). In his "**·6**" game, any one counter can be removed, except an R_1 (= a single counter standing on its own). The $G(R_x)$ series ($x = 1, 2, \ldots$) is a waltz (N.B. some notes span two bars):

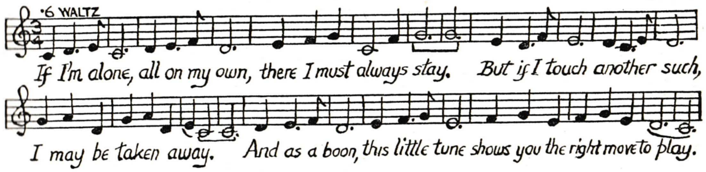

FIG. 2.

But at that point the tune completely broke down. I asked Guy if he could think of any reason for that: he said, "Yes, an error I made in the calculation." After correction the waltz proceeds:

FIG. 3.

This tries to be periodic with period 26, but jumps keep appearing.

Many other such games give tuneful, somewhat periodic series, for no evident reason. Guy discovered two curious exceptions: his "**·4**," remove 1 counter not at the end of a row, has exact period 34 for $x \geqslant 54$, and Kayles, remove 1 or 2 adjacent counters, has exact period 12 for $x \geqslant 71$.* Thus these games have a complete analysis. But generally it might be helpful to bring in a professional musician to study number theory. Perhaps a thorough study of Fermat's Last Theorem will uncover the Lost Chord. After all, why not?

Integration

Comparison of the results of the recent *Eureka* questionnaire with those recorded for a similar questionnaire published in 1939 reveal no startling changes in the tastes and habits of mathematicians. The traditional judgements that were false about us then are false about us now. We have no especial predilection for Bach—indeed most of us prefer Beethoven. Our "Alice in Wonderland's" gather as much dust on our shelves as other men's. Bridge and chess are not our only recreations.

We are here, it seems, because of our own original merit and not through inherited talent. Few of us can find indications of mathematical ability in our forebears.

It is pleasing to note that we are still addicted to nonsense and gaiety. We read nonsense verse although our taste for other poetry is slight and ill-defined. We applaud Gilbert and Sullivan almost to a man.

An observer given a birdseye view of Cambridge mathematicians at work would find plenty to entertain him. The nervous energy and tension created by our intellectual struggles finds many outlets. We talk, chew our nails and our handkerchiefs, cry out, twist our hair, drum our fingers on the desk in impotent frenzy, lie on the floor. One of us even strokes his "fine R.A.F.-type moustache." These little habits do not occupy too much of our time for we are only moderately industrious. One of us claims to work forty hours a week, but for most of us half that number is nearer the mark. Our general opinion of the value of lectures and supervisions is high and we are satisfied with our lot and would not change it for another subject.

* Further discoveries: m-plicate Kayles, removing m or $m + 1$ consecutive counters has period $12\,m$ for $x \geqslant 71m$, and m-plicate double Kayles, removing from m to $(3m + 1)$ consecutive counters, has period $24m$ for $x \geqslant 142m$. The reasons are only partly understood: I am informed that a fuller discussion is being prepared for publication. Meanwhile I have Guy's consent to quote the results of his calculations.

The answers to the question of a favourite lecturer showed that many have claims to popularity. Perhaps the most heartfelt reply was the one that said "myself!"

Our relaxations are many and varied. We are moderately sporty but only those of us who are women, it seems, find time to row. The most popular relaxation is sleeping.

The question that evoked the most lively and varied answers was the last. Perhaps the aptest was that it would give the Editor some entertainment. It did!

J. C. P.

Solutions to Problems Drive

1. Figure 1 is the simplest solution.
2. 13; 21. (Derive difference equation $u_{n+1} = u_n + u_{n-1}$.)
4. Charles. (Bryan takes less than 35 sec. by triangle inequality, Charles takes more than 35 sec. since one of the angles CRB and CJB is obtuse.)

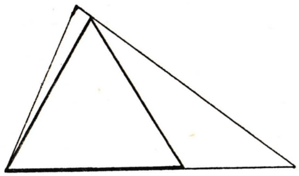

FIG. 1.

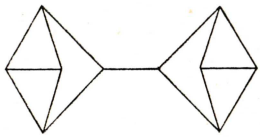

FIG. 2.

5. (*a*) 282824.
   (*b*) 282841. (Derivation of $\sqrt{2}$.)
6. "Honesty is the best policy."
7. See Figure 2.
8. $\exists \leqslant 16$ plots. $13 \supset$ magpies. $1 \supset$ gardener.
   $\therefore \exists \leqslant 2$ other plots adjacent to plot $\supset$ gardener.
   $\therefore 1$ side of plot $\supset$ gardener is a garden boundary.
9. £8 may be paid as 12 4/– pieces, 1 half-crown, 1 crown and 3 30/– pieces.

   £100 may be paid by 135 coins, as 84 4/– pieces, 49 30/– pieces, 1 10/– piece and 1 half-crown. (4/- pieces which become worth 4/4¾d. are needed for all payments not a multiple of 2/9d.)
10. (*a*) 3 weights, 1 oz., 4 oz., 16 oz.
    (*b*) With 6 weights, 1 oz., 4 oz., 1 lb., 4 lb., 16 lb., 64 lb., we can weigh any weight up to 12 stone 2 lb. 10 oz.

Book Reviews

Inequalities. By HARDY, LITTLEWOOD and POLYA. (Second Edition.) (Cambridge University Press.) 27s. 6d.

Exact knowledge is a dream of the past—an ideal with which we are no longer seriously concerned. For the last half-century mathematics has been concerned with inequalities. It is therefore remarkable that this is the only book yet written which deals with them systematically.

The first six chapters deal with the fundamental inequalities—Schwarz, Hölder and Minkowski in their various disguises. They are treated first for finite sums, then for infinite series and finally for integrals. This involves a good deal of repetition, particularly since so much emphasis is laid on "appropriate" methods of proof; in spite of many fascinating asides, these chapters drag a little for the casual reader. But one could scarcely turn to them and fail to find exactly what one wanted.

The remaining chapters are four essays on the calculus of variations, bilinear forms, Hilbert's inequality and rearrangements. These are written with a lighter hand and could not be improved on; the only thing to regret is that there are not a dozen of them. In particular, they have the rarest of all virtues in a mathematical book—they do not suggest that the subject is complete and there is nothing more of interest to be discovered in it.

The changes from the first edition are small, at the insistence of the printers; but there are three interesting appendices. If you have the first edition, you can read them in any respectable book-shop. If not, buy this at once. It is no use waiting for another book on the subject, for no sensible writer will try to compete with this one. P. S-D.

Methods of Algebraic Geometry, Volume Two. By W. V. D. HODGE and D. PEDOE. (Cambridge University Press.) 42s.

This is the second of three volumes designed to provide a convenient account of the modern algebraic methods available to geometers. The first volume, which appeared in 1947, contained an account of the basic properties of projective space of n dimensions, preceded by a section devoted to pure algebra. In the later chapters the authors limited themselves to geometry over fields without characteristic, and this restriction is continued throughout the present volume. Thus a sound algebraic basis is provided for the classical case of geometry over the field of complex numbers and a wider range of methods discussed than would otherwise have been possible. Also, the student with previous experience only of classical geometry will find the accounts of geometry over fields of finite characteristic easier to grasp after he has mastered the present work, in which the general pattern of classical geometry is retained.

The first three chapters deal with the general theory of algebraic varieties in projective space. Much use is made of the "Zugeordnete Form" of Chow and van der Waerden, which is developed in the first chapter. The authors then discuss the foundations of algebraic correspondences, leading to the idea of multiplicity. The third chapter is devoted to the intersections of varieties and the algebraic theory of systems of varieties. The latter part of the book is taken up with the development of the properties of quadrics and Grassmann varieties by the strictly algebraic methods described earlier.

The writing throughout maintains the very high standard of the first volume. The reader will find this a most attractive work, particularly if he already has a slight acquaintance with the "Zugeordnete Form."

M. A. H.

The Lebesgue Integral. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 40.) By J. C. BURKILL. (Cambridge University Press.) 12s. 6d.

As an undergraduate, I attended Dr. Burkill's Tripos lectures on Convergence. I thought then, and I would not disclaim the opinion now, that their only fault was that they made the subject appear so simple that one could absorb it without overmuch attention or thought—an illusion which was soon dispelled by examples. The tract under review gives a similar impression of removing the last trace of mystery from one of the analyst's finest tools—the Lebesgue integral—and since a book, unlike lectures, can be referred to again as soon as the feeling of mastery begins to fade, this is almost wholly to the good. Moreover, a number of theoretical examples are included, together with notes on their solution: this is a commendable practice which should enable the reader both to test and to extend his understanding of the subject-matter.

The range of the book is surprisingly wide for its physical volume; it includes all the main properties of the Lebesgue integral, with such results from point-set theory as are required, as well as some of the differential properties of arbitrary functions, and a useful introduction to the Lebesgue-Stieltjes integral. This compression has been achieved, neither by unduly condensed proofs nor by assuming overmuch preliminary knowledge, but by a careful choice of the simplest proof of each theorem. Only in one place do I feel that the argument seriously needs expansion, and this unfortunately is in section 2.2 where the foundations of the theory of measure are being discussed. "Taking the limit: we have the general case" seems to me to conceal a not very obvious argument which, on account of the fundamental importance of the result, should have been given in full. Associated with this, there seems to be a little confusion in section 2.4, where a proof is given of a result (*a*) which appears obvious from the definition in 2.3, while (*b*), the less obvious result, is dismissed as "similar." Another more general criticism is that I should like to see more indication, perhaps in footnotes, of the complications that appear in some proofs and theorems, particularly those involving differentiation, when we pass to more than one independent variable: the author may well have felt that such indications were out of place in a work of this character.

These are, however, minor criticisms of an excellent tract which will no doubt enjoy for a considerable time a deserved place as a standard English text on this important subject, suitable for any competent mathematician of "Part II Tripos" attainments. A. J. W.

An Introduction to Modern Prime Number Theory. By T. ESTERMANN. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 41.) (Cambridge University Press.) 12s. 6d.

This tract gives an account of some of the recent developments in prime number theory. In particular it treats the theory of the number of primes in a given arithmetic progression; it also contains a proof of the theorem of Vinogradoff concerning Goldbach's conjecture.

Vinogradoff proved in 1937 that every sufficiently large odd number could be represented as the sum of three prime numbers; he used a method that had been developed by Hardy and Littlewood. Goldbach's conjecture that every even number is representable as the sum of two primes remains unproved. The tract is probably the first account of Vinogradoff's theorem in book form in English.

Unfortunately there is no space in the tract for the treatment of various other developments in the theory of prime numbers. In particular no account of the recent "elementary" proof of the prime number theorem due to Erdös and Selberg is given. Nor is any account given of the various results on the difference between consecutive prime numbers.

It is stated in the introduction to the tract that it is intended for "those mathematicians who are not specialists in the theory of numbers" to enable them "to learn some of its non-elementary results without too great an effort." The tract is written in a style which makes most of the mathematical arguments as elementary as possible but this may make it difficult for the reader to follow the general line of argument as it is obscured by the mass of detail. Although it is nice to know the numerical values of some of the constants, these values do tend to spoil the appearance of the formulae and make the tract less readable. Further, the constants given have little computational value as they are far from "best possible".

There is a useful table of theorems and formulae at the end of the tract, but the index is very short and there is no bibliography. A few references are given in the introduction. The tract is well printed and the reviewer has not detected any misprints. C. B. H.

Cosmology. By H. Bondi. (Cambridge University Press.) 22s. 6d

Cosmology is usually regarded as an obscure and fruitless subject which can safely be ignored by real scientists. Mr. Bondi's book should do much to dispel this attitude, for it is written in a clear and attractive way, and it presents a convincing case for the relevance of cosmology to both astrophysics and fundamental physical thinking.

The book contains a well-balanced introduction to the various rival theories, and to the relevant observational data. The amount of detailed mathematics is kept to a minimum, so that the aims, methods, and results of each theory can readily be grasped and compared.

In addition, the book has three features which give it special interest. First, there is the emphasis on the fact that cosmology is a branch of physics in its own right, and is not, as often supposed, a minor application of general relativity.

Second, there is a detailed discussion of the possibility that there may be local phenomena which are largely determined by distant matter. This possibility arises because the great bulk of distant matter may outweigh its great distance. Hence observations of local phenomena may furnish important information about the structure of the universe as a whole. For instance, the fact that it is dark at night, in conjunction with other astrophysical data, enables one to deduce that distant galaxies are rapidly receding—a deduction which can be independently confirmed by observations of the red-shifts in the spectra of these galaxies. A more speculative example of this idea is contained in the discussion of Mach's principle (according to which the inertia of matter arises from the totality of masses in the universe). This principle inspired the discovery of general relativity, but is not given complete expression in that theory. Thus it is still uncertain whether or not inertia is a manifestation of a strong coupling between local and distant matter.

Finally, a detailed discussion is given of the steady-state model of the universe, proposed by Bondi and Gold in 1948. In this model the continual creation of matter maintains a constant mean density in the universe despite the expansion. An account is also given of Hoyle's modification of general relativity to allow for this creation process. The steady-state model has many attractive features, but the available evidence is not decisive in distinguishing between it and the classical Lemaitre model. However, as Mr. Bondi points out, our understanding of astrophysics may soon develop sufficiently for crucial evidence to be discovered. D. W. S.

Mathematics by the Fireside. By G. L. S. SHACKLE. (Cambridge University Press.) 15s.

The object of this book is to teach the mathematically unlearned something about the fundamental concepts of mathematics. This Herculean task is to be achieved by their overhearing conversations between two children, George and Lucy, and George's father. It is unfortunate, but almost inevitable, that these three characters should prove unattractive. The father is a bore without a spark of humanity. He uses the cutting of the cake at a birthday party as an excuse for a lecture on rational numbers. It is a tribute to the exasperating intelligence of George and Lucy that this birthday treat is received not with contempt but with enthusiasm.

However, characters like these must be accepted as conventions in a book of this type. Professor Shackle has enlivened his book with imaginative ingenuities of exposition. Thus mathematical induction becomes "building taller and taller towers" and complex numbers make their appearance under the thrilling guise of finding pirate gold. The reviewer is unable to say if the book will achieve its avowed object, but it will entertain any mathematician who reads it. J. C. P.

Mathematics, Queen and Servant of Science. By E. T. BELL. (G. Bell & Sons, Ltd.) 21s.

This volume by Professor E. T. Bell of the Californian Institute of Technology is presented as a thorough revision and very considerable amplification of two earlier works, *The Queen of the Sciences* published in 1931, and *The Handmaiden of the Sciences*, published in 1937. Thus the title indicates that on reflection Professor Bell wishes to emphasise the essential unity of all science, but it seems a pity that the handmaiden has been forsaken for a mere servant.

The principal theme of the book is that much pure mathematics, disclosed by such means as weakening the postulates defining a known system, and then studied intensively for its sheer beauty by pure mathematicians, has turned out, often unexpectedly, to be of fundamental importance in the other sciences, particularly physics. This theme is unfolded discursively by means of many interesting examples which are usually well explained. Those who have just left school and intend to read mathematics will find many vistas opened up to attract them onwards, and will remember the counsel of Abel to study the masters, not the pupils. Non-mathematicians—and the present reviewer is gratified to note that lawyers are the first named by Professor Bell in his list of the classes of laymen whom his works have already reached—will surely be liberated from their common misconception that the higher mathematics is the coldest of intellectual heights.

The frequency of the refrain "which will be explained later" in the earlier chapters of the book; the occurrence of such a sentence as "Associative algebras do not exhaust either linear algebras or linear associative algebras"; and the sacrilegious designation of the great master of English Science as I. Newton (1642 to 1727) are minor blemishes. In addition the general impression left would be deepened if the serious ethical problem facing mathematicians as a result of the application of scientific discoveries to wholesale massacre were treated more seriously, instead of being dismissed by an expression of the belief that possibly it is too late to ban quadratics even for the Eskimos.

J. P. H. M.

The following book has been received and will be reviewed in our next issue:

The Spirit of Applied Mathematics. By C. A. Coulson. (Oxford University Press.)

The Mathematical Association

President: Professor T. A. A. Broadbent.

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching,* aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

The subscription to the Association is 21s. per annum: to encourage students, and those who have recently completed their training, the rules of the Association provide for junior membership for a limited period at an annual subscription of 10s. 6d. Full particulars can be had from The Mathematical Association, Gordon House, Gordon Square, London, W.C.1.

The *Mathematical Gazette* is the journal of the Association. It is published four times a year and deals with mathematical topics of general interest.