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Nim-Arithmetic

MANY people are familiar with the theory of the game of Nim. This uses an unusual type of "addition" of two numbers, the Nim-sum, $x\,(+)\,y$, of x and y being formed by writing out the binary expressions of x and y and adding the columns mod. 2. For example:

$$26\ (+)\ 3 \quad \begin{array}{r} 11010 \\ 11 \\ \hline 11001 \end{array} \text{ is } 25; \quad 45\ (+)\ 31 \quad \begin{array}{r} 101101 \\ 11111 \\ \hline 110010 \end{array} \text{ is } 50.$$

This can of course be taken as an operation on ordinary integers, but it has a simpler aspect.

NUMBERS AS POLYNOMIALS.

Consider the field of residues mod. 2; that is, the set of two numbers, 0 and 1, with $1\ (+)\ 1 = 0$, and the other laws as usual. We may now take polynomials over this field in an indeterminate x. Alternatively, we may say: 2 is not in the base field, so it will cause no confusion to write 2 for x. We then have polynomials in 2, which may be taken as the ordinary binary representations of ordinary numbers. From this attitude, we may define the Nim-product of two numbers as the number obtained by ordinary polynomial multiplication. It now follows that the ordinary commutative, associative and distributive laws of algebra hold, that there are no divisors of zero, and that our polynomials (non-negative integers) form an integral domain. Also that unique factorization holds, that is, each number is uniquely expressible as a product of certain Nim-primes, which are those elements having no factors except themselves and unity.

MULTIPLICATION.

The purpose of this article from now on is to treat the polynomials as ordinary numbers, and to develop arithmetic in the usual way. We must first give a mechanical method of multiplication. This is best derived from the distributive law, and the observation that multiplication by powers of 2 is the same as in ordinary arithmetic. Thus, for example:

$$11\ (\times)\ 15 = (8\ (+)\ 2\ (+)\ 1)\ (\times)\ 15 = 120\ (+)\ 30\ (+)\ 15 = 120\ (+)\ 17 = 105$$

$$13\ (\times)\ 15 = (8\ (+)\ 4\ (+)\ 1)\ (\times)\ 15 = 120\ (+)\ 60\ (+)\ 15 = 68\ (+)\ 15 = 75.$$

A multiplication table can be constructed, even numbers being omitted since multiplication by 2 is as usual.

1	3	5	7	9	11	13	15
3	5	15	9	27	29	23	17
5	15	17	27	45	39	57	51
7	9	27	21	63	49	35	45
9	27	45	63	65	83	101	119
11	29	39	49	83	69	127	105
13	23	57	35	101	127	81	75
15	17	51	45	119	105	75	85

The main regularity is that the product of two polynomials has the sum of their degrees. (In fact, classification into degrees is the only really useful ordering of Nim-numbers.) It follows that 2 and 3 each divide half the numbers of any given degree; 4, 5, 6 and 7, each a quarter, etc.

SUBTRACTION.

We note as a direct corollary to the definition of Nim-addition that $x\ (+)\ x = 0$. For in each column, we form a mod 2 sum $0\ (+)\ 0$ or $1\ (+)\ 1$, giving 0 in each case. It follows that the solution "$b\ (-)\ a$" of $a\ (+)\ x = b$ is $x = a\ (+)\ b$, and so subtraction is the same as addition. One more property of the same kind may be noted.

$$\begin{aligned}(x\ (+)\ y)^2 &= x\ (\times)\ (x\ (+)\ y)\ (+)\ y\ (\times)\ (x\ (+)\ y),\\ &= x\ (\times)\ x\ (+)\ \{x\ (\times)\ y\ (+)\ x\ (\times)\ y\}\ (+)\ y\ (\times)\ y,\\ &= x^2\ (+)\ y^2,\end{aligned}$$

a law which has been misapplied to ordinary arithmetic; this is its home!

DIVISION.

The process is effected by the division algorithm, which has the same properties as in the case of ordinary polynomials over the field of rationals. We also need tests for divisibility. With 2 the test is as usual.

Theorem.—A number is divisible by 3 if and only if it has an even number of 1's in its binary representation.

(1) $3\ (\times)\ x = 2x\ (+)\ x =$ the Nim-sum of two numbers each with the same number of 1's. An even number of these cancel out in Nim-addition, leaving us with an even number.

(2) An even number of 1's can be paired off, and we see by the distributive law that we need only to prove that 3 divides each of the numbers (containing only two 1's) obtained by this

pairing. Removing a superfluous factor of 2, we have a number $2^r + 1 = 3 (\times) (2^r - 1)$ as is easily verified.

In fact we always use the binary representation, and since, as we shall prove later, each odd Nim-prime divides $2^r - 1$ for some r, we form the number into blocks of r, add these Nim-wise, and test the resulting number (of degree $< r$) by ordinary Nim-division.

PRIMES.

These are found by the usual "sieve" method; in fact, we test each number for divisibility by known primes of $\leqslant \frac{1}{2}$ its degree (corresponding to the usual $\leqslant$ its square root). Apart from 2, we need only test odd numbers.

Degree 1: 2, 3 are prime.

Degree 2: we have 5, 7. 3 divides 5 only, so 7 is prime.

Degree 3: of 9, 11, 13, 15; 3 divides 9 and 15, so that 11 and 13 are prime.

Degree 4: of 17, 19, 21, 23, 25, 27, 29, 31; 3 divides 17, 23, 27, 29; 7 divides 21, 27, so that 19, 25, 31 are prime. And so on.

DISTRIBUTION OF PRIMES.

This is a central problem in the Theory of Numbers, and it is natural to consider it here. Our resources, however, are greater than those of the number-theoreticians, and we have an exact formula.

Define N_n as the number of primes of degree n, and write $M_n = n\, N_n$.

Lemma.—$2^n = \sum_{d|n} M_d$, where the summation is over all (ordinary) divisors of n (including n and 1).

The following results on finite fields are needed:—

(1) The number of elements of a finite field is a power p^n of its characteristic; all ^{n} are possible.

(2) Any two finite fields of p^n elements are isomorphic, being the root field of $x^{p^n} = x$ over the prime field GF(p^n).

(3) The multiplicative group of all nonzero elements is cyclic.

We consider algebraic extensions of the field of residues mod 2 GF(2); all have characteristic $p = 2$. Let f be a Nim-prime (irreducible polynomial) of degree n. If we adjoin one of its roots to GF(2) it is clear that all elements of the field it generates are of the form

$$a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_{n-1}\,\alpha^{n-1};$$

the a_i are 0 or 1. Thus there are 2^n such elements; we have GF(2^n)

and α satisfies $\alpha^{2^n} = \alpha$. In the root field of f, all the roots of f satisfy this equation and therefore belong to its root field, GF(2^n). Hence this is the root field of f. Since all the roots of $f = 0$ satisfy $x^{2^n} (+) x = 0$, f divides this polynomial: either $f \equiv x$ or f divides $x^{2^d-1} (+) 1$. If g is irreducible, of degree d/n, then it divides $x^{2^n-1} (+) 1$. But since d divides n, $2^d - 1$ divides $2^n - 1$, and so $x^{2^d-1} (+) 1$ divides $x^{2^n-1} (+) 1$ and therefore all roots of $g = 0$ satisfy $x^{2^n} (+) 1 = 0$. Hence they also belong to GF(2^n). So, trivially, do roots of $x = 0$.

We have proved that all $\sum\limits_{d/n} M_d$ roots of irreducible polynomials of degrees dividing n belong to GF(2^n), a field with 2^n elements. If we can prove the converse, the lemma is established.

Let a be any element of GF(2^n), and let it satisfy the irreducible equation $f(x) = 0$ of degree r. The root field of $f(x)$ is GF(2^r). The multiplicative group of this is cyclic, of order $2^r - 1$. Now, since a belongs to GF(2^n), so do all roots of $f(x) = 0$, since they are linear combinations of powers of a with coefficients 0 and 1, as we have proved above. Hence so does every element of GF(2^r). In particular, the multiplicative group of nonzero elements contains as a subgroup the cyclic group of $2^r - 1$. Hence, by Lagrange's theorem, $2^r - 1$ divides $2^n - 1$. If $n = ar + b$, $b < r$, $2^r - 1$ divides $2^{ar} - 1$ and therefore $2^n - 2^{ar}$, and therefore $2^b - 1$. But $b < r$, so $b = 0$ and r divides n. Q.E.D.

Theorem.—$M_n = 2^n - \sum\limits_{p/n} 2^{n/p} + \sum\limits_{p,q/n} 2^{n/p,q} - \ldots = nN_n$, where

where the first summation is over all primes, p, dividing n, and the second over all pairs of distinct primes dividing n.

The proof is by an induction argument, of which the first two steps (covering $n < 30$) will be given. If n is a prime, p^r,

$$2^n = \sum_{0 \leqslant i \leqslant r} M_{p^i}; \quad 2^{n/p} = 2^{p^{r-1}}$$

Subtracting,

$$2^n - 2^{n/p} = M_{p^r} = M_n.$$

If n has only two distinct prime factors, $n = p^r q^s$, we have

$$2^n = \sum_{0 \leqslant i \leqslant r} \sum_{0 \leqslant j \leqslant s} M_{p^i q^i},$$

with corresponding expressions for $2^{n/p}$, $2^{n/q}$, $2^{n/pq}$.

Adding and subtracting,

$$2^n - 2^{n/p} - 2^{n/q} + 2^{n/pq} = M_{p^r q^s} = M_n.$$

Similarly for the other cases.

DEDUCTIONS FROM THE PRIME NUMBER THEOREM.

First, numerical deductions; we can tabulate numbers of primes of different degrees as follows:—

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_n	2	1	2	3	6	9	18	30	56	99	186	335	620	1161

Also $$n\,N_n = 2^n + O\,(2^{\frac{1}{2}n}).$$

For the number of terms in the expression for $n\,N_n$ is certainly not more than 2^t, t being the number of primes $< n$, $t \sim n/\log n$.

Thus $$\begin{aligned} nN_n &= 2^n - 2^{\frac{1}{2}n} + O\,\{2^t\,(2^{n/3})\},\\ &= 2^n + O\,(2^{\frac{1}{2}n}) + O\,\{2^{n(\frac{1}{3}+1/\log n)}\},\\ &= 2^n + O\,(2^{\frac{1}{2}n}).\end{aligned}$$

The number of Nim-primes $< 2^n$ is the number of degree $\leqslant n - 1$ which is $$\sum_{r=1}^{n-1}\left\{\frac{2^r + O\,(2^{r/2})}{r}\right\} = (1/n)\,2^n + O\,(2^{n/2}).$$

Now the number of ordinary primes $< 2^n$ is asymptotic to $2^n/\log\,(2^n) = 2^n/(n\log 2)$. Hence there are about $\log 2 = \cdot 693$ times as many Nim-primes as ordinary primes in such a range.

C. T. C. WALL.

Diophantus Smith

When the will of Diophantus Smith (R.I.P.) was examined, it was found that he had directed that £100 should be divided among his closest 100 friends. Moreover, each man was to recive £10, each woman 10s., and each child 2s. 6d. The friends of the great man made no objection to being placed in order of closeness; this procedure was performed without difficulty, and three such lots were drawn up, for men, women and children respectively. The executors' headache was to decide how many from each list were to benefit.

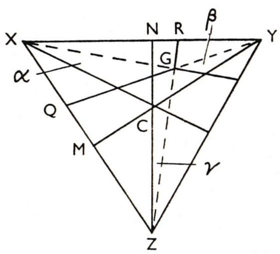
How many did?

(*Solution on page* 22)

A Theory of Biased Dice

Most people are aware that the behaviour of dice can be influenced by suitably biasing them with weights, and that even the numbering of the faces by making small indentations favours higher numbers turning up. We here attempt to work out a simple theory of the effects of all kinds of bias. We consider the case of a level plane and make the following assumptions:—

(1) That the die finally settles by falling on one corner and continues with the centre of gravity of the die moving in the plane of the vertical through the corner on which the die lands. Then one edge strikes the plane, and the face which falls down is the face on the same side of the edge as the centre of gravity.



(2) Until the beginning of this final stage, the motion is so violent that there is complete chaos, so that (*a*) all corners are equally likely to be the ones involved as described; (*b*) for any one corner, all possible orientations of the die relative to the vertical are equally likely as measured by element solid angle.

Clearly, these assumptions are realistic for a surface of glue, but this is not useful in general practice or practicable for experiment. Systematic inertia effects are neglected.

Suppose the corner O is the one we must consider. The problem is best solved by a geometrical representation, using areas of spherical triangles. We represent all possible positions of the vertical OV relative to the triad O*xyz* (O*x*, etc., along edge of die) by the spherical triangle XYZ bounded by this triad on a unit sphere of centre O. The behaviour then becomes self-evident.

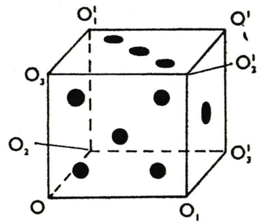
If V lies in the spherical quadrilateral GRXQ then the face OYZ will fall, and so on for the other faces. Routine trigonometry gives the area of this: neglecting second order terms with α, β, γ small, we have

$$\text{GRXQ} = \tfrac{1}{3}\,\text{XYZ} + \frac{\gamma - \beta}{\sqrt{3}}.$$

Thus, given the corner O, the respective chances for faces OYZ, etc., are

$$\tfrac{1}{3} + \frac{2}{\pi\sqrt{3}}(\gamma - \beta), \text{ etc.} \qquad \ldots \qquad \ldots \qquad (1)$$

$(\text{XYZ} = \pi/2)$.



We now suppose that the geometrical centre of the die has co-ordinates

$$l + l\text{A},\ l + l\text{B},\ l + l\text{C};$$

$(\text{A} = \text{B} = \text{C} = 0$ if the die is perfectly cubical) and the centre of gravity has co-ordinates

$$l + l\text{A} + la,\ l + l\text{B} + lb,\ l + l\text{C} + lc;$$

$(a = b = c = o$ if the die is perfectly uniform), all relative to the triad $Oxyz$. Then we approximate immediately:

$$\gamma = -\frac{(A - B) + (a - b)}{\sqrt{2}}, \text{ etc.},$$

and substitute in (1), giving

$$\tfrac{1}{3} - \frac{\sqrt{6}}{\pi}\left[(\text{A} - \text{S}) + (a - s)\right], \qquad \ldots \qquad \ldots \qquad (2)$$

where $\quad \text{S} = \tfrac{1}{3}(\text{A} + \text{B} + \text{C}) \qquad s = \tfrac{1}{3}(a + b + c)$.

We now examine a die in particular, with the customary opposite faces adding up to seven. Relative to these different triads $O\,(x, y, z)$, $O_1\,(x, y, z)$, etc., the characteristics of the die are different. For example, using a natural extension of our notation,

$$a_1 = -a,\ s_1 = \tfrac{1}{3}(-a + b + c), \text{ etc.}$$

Now, a 1 coming up can be caused by any of

$$\left.\begin{array}{lll} 1. & O & \text{corner—YZ face down,} \\ 2. & O_2 & \text{corner—YZ face down,} \\ 3. & O_3 & \text{corner—YZ face down,} \\ 4. & O'_1 & \text{corner—YZ face down.} \end{array}\right\}$$

Using (2) and our basic assumptions again, that each corner may be allotted a probablity $\frac{1}{8}$, we write down the chance that 1 should turn up:

$$p(1) = \tfrac{1}{8}\left\{\left(\tfrac{1}{3} - \frac{\sqrt{6}}{\pi}[A - S + a - s]\right)\right.$$
$$\left. + \left(\tfrac{1}{3} - \frac{\sqrt{6}}{\pi}[A_2 - S_2 + a_2 - s_2]\right) + \ldots\right\},$$

whence

$$p(1) = \tfrac{1}{6} - \frac{\sqrt{6}}{2\pi}(A - S) - \frac{\sqrt{6}}{3\pi}a,$$

$$p(6) = \tfrac{1}{6} - \frac{\sqrt{6}}{2\pi}(A - S) + \frac{\sqrt{6}}{3\pi}a, \text{ etc.}$$

An experiment was performed with six dice. These were divided into two sets of three with the following average characteristics:—

	$(A - S)\times 10^3$	$(B - S)\times 10^3$	$(C - S)\times 10^3$	$a \times 10^3$	$b \times 10^3$	$c \times 10^3$
1.	$-8{\cdot}0$	$+8{\cdot}6$	$-0{\cdot}6$	$+11{\cdot}4$	$+3{\cdot}3$	$+4{\cdot}0$
2.	$+1{\cdot}0$	$-5{\cdot}7$	$+4{\cdot}7$	$+11{\cdot}1$	$+4{\cdot}4$	$+0{\cdot}3$

A, B and C were measured with a micrometer screw gauge and a, b, c measured by assuming that this effect was purely due to the holes used to mark the numbers. They were filled with clay and measured for each face. The results were obtained with each of the sets of three dice, each die being tossed 1,000 times, making a total of 3,000 times for each set. The values of (frequency $- \frac{1}{6}$) $\times 10^3$ are tabulated below:

	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(4)$	$f(6)$
1.	$-4{\cdot}6$	$-13{\cdot}0$	$+13{\cdot}6$	$-5{\cdot}6$	$-17{\cdot}3$	$+27{\cdot}0$
2.	$-11{\cdot}0$	$-6{\cdot}0$	$-7{\cdot}6$	$+5{\cdot}0$	$+13{\cdot}0$	$+6{\cdot}6$

Each of these has a standard deviation of about 7·0, but by combining them suitably, we can obtain some fairly significant conclusions.

We first test the weight bias effect. The geometrical bias is eliminated by considering in quantities $x = f(6) - f(1)$, $y = f(5) - f(2)$, $z = f(4) - f(3)$, and considering the coefficient m for the most likely direct proportional relationship between these quantities, and a, b, c respectively, which our theory would anticipate as being approximately

$$2\frac{\sqrt{6}}{3\pi} = 0{\cdot}52$$

Now by the method of least squares,

$$m = \frac{ax + by + cz}{a^2 + b^2 + c^2}$$

Assuming standard deviation of x, y, $z = \sigma \approx 10$, we obtain that the standard deviation of m

$$\sigma\,(m) = \frac{\sigma}{\sqrt{(a^2 + b^2 + c^2.)}}$$

Using this method, we find that in the two cases:

$$1. \quad m_1 = 1{\cdot}7 \qquad 2. \quad m_2 = 1{\cdot}9$$
$$\sigma\,(m_1) = 0{\cdot}8 \qquad \sigma\,(m_2) = 0{\cdot}8$$

We can calculate the geometrical bias similarly by considering $X = f\,(6) + f\,(1)$, etc. Then correspondingly

$$1. \quad M_1 = 3{\cdot}0 \qquad 2. \quad M_2 = 1{\cdot}0$$
$$\sigma\,(M_1) = 0{\cdot}8 \qquad \sigma\,(M_2) = 1{\cdot}3$$

compared with the predicted values of m, $M = 0{\cdot}78$.

Although errors are large, the results seem to suggest fairly significantly that the predicted values are too low, something of the order of a third of the experimental values. Assumption (2*a*) would appear to be responsible for a large part of this discrepancy. It is hoped that further work on this problem will enable some account to be made of the effect of the bias upon the movement of the die before its final stage.

J. D. Roberts.

Plasticity

STUDENTS of school physics first learn about elasticity when they hang various weights on a copper wire and measure the corresponding extensions of the wire. If the students are careful, they draw a stress-strain graph similar to the one in their textbook; the strain increases linearly with stress until the elastic limit and then increases faster and faster until the yield point where the wire gives completely. The textbook quotes Hooke's law "ut tensio sic vis," to describe the linear part of the graph, and disposes of the rest in a qualitative sentence or two. This neglect of the non-linearity of the stress-strain relation continues in the Mathematical Tripos, for which Hooke's law is neatly generalised in tensor form on the convenient hypotheses that the material is isotropic and the strain small. Thus three basic assumptions are made—isotropy, small strain and linearity. The first is dropped in the theory of anisotropic elasticity, the second in the theory of finite strain, and the third in the theory of plasticity, the subject of this discussion.

The theory of elasticity gives a reliable approximation to the behaviour of materials under stress provided the stress is not too large. The theory of plasticity gives a further approximation when the stress is large. We define the mathematical model of a perfectly plastic solid. This solid is described by the generalised Hooke's law except at those points where the stress components are large enough to satisfy a certain condition, called the yield criterion. Where the yield criterion is satisfied a new plastic stress-strain relation is used. A number of yield criteria, varying from the mathematically simple but physically inaccurate to physically accurate but mathematically complex, have been tried. A similar conflict arises in the choice of plastic stress-strain relation. However, a reasonable compromise between the mathematical and physical requirements can be found for both the yield criterion and plastic stress-strain relation.

The derivation of the yield criterion and plastic stress-strain relation is only the beginning of the difficulties. Even in elasticity it is possible to give exact solutions to only a few boundary problems. The general problem of plasticity comprises a plastic problem in the region where the yield criterion is satisfied as well as an elastic problem in the region where the yield criterion is not satisfied. These regions have an unknown frontier on which the boundary conditions are unknown.

Only fragments of this chaos have been resolved. A number of techniques have been developed for a few limited types of problem. A known elastic solution shows where plastic strain begins and often indicates the subsequent nature of the elastic-plastic solution. Also intelligent guesses which lead to solutions may be inspired by other observations such as those of symmetry, particular cases or experiment. The last resort is the multitude of approximation methods available. Apart from direct numerical approximations, it is possible to use extremum principles (theorems of maximum and minimum energy) which are generalisations of those of elasticity. For two-dimensional problems, some quite powerful methods have been developed.

Why are these dire mathematical difficulties created? Processes of metal forming, e.g. rolling, drawing and extrusion, require a knowledge of the stress and strains involved. The theory of plasticity is specifically designed to describe the distortion of metals both because they are more important in engineering than other ductile solids and because their properties are better documented. Stimulated by the demand for cold-worked metals such as steel, plasticity is a rapidly expanding subject of both theoretical and practical interest.

P. DRAZIN.

Mathematician Ted
Fell ded;
Qed.

CLERIHEW

Prize Contributions

PRIZES of book tokens have been awarded to C. T. C. Wall and J. D. Roberts for their contributions published in this issue of *Eureka*. It is proposed to offer further prizes in 1956 for contributions submitted under the following conditions:—

(1) At present, only Cambridge undergraduates will be eligible to compete.

(2) Any contributions will be considered, except Book Reviews, Problems Drives and Society Reports. They can be grave or gay, and should preferably have some mathematical flavour.

(3) Prizes will take the form of book tokens. The total prize money will not be more than £5, and will be divided at the absolute discretion of the Editorial Committee, whose decision is final.

(4) Acceptance of a contribution for publication will not necessarily be regarded as an indication that the contributor should receive a prize.

The Editor would be appreciative if contributions could be typed in double spacing on one side of the paper, or written clearly and intelligibly. Monstrosities of English and unnecessary abbreviations should be avoided. Contributions can be sent to the Editor, c/o The Art School, Bene't Street, who trusts that the mailbag will be a full one!

By Induction or Ever Been Had?

(i) Punctuate the following:—

John where Willie had had had had had had had had had had had full marks.

(ii) Show that there exist intelligible sentences containing (14.3^n-3) successive *had*'s, where n is any non-negative integer. (Solution on page 23.)

Problems Drive

The problems drive is a competition conducted annually by the Archimedeans. Competitors work in pairs and are allowed five minutes per question; the winners, besides receiving a prize, are invited to set the questions for the following year.

1. In the Inca civilisation, arithmetic was the same as it is to-day, and mathematicians counted to the base 10 and used the same symbols (1, 2, . . . etc.) as we do, but each symbol denoted a different digit. We have just three equations true in their arithmetic, namely,

$$2 \times 2 \times 6 = 24,$$
$$5 \times 6 = 30,$$
$$5 \times 8 + 7 + 1 = 48.$$

What does the symbol 9 represent to the Incas?
What is their answer to the question: Evaluate 2^3?

2. A function $f(a, b, c, d)$ is defined with these two properties:—

$$f(a, b, c, d) + f(b, c, d, e) + f(c, d, e, a) = 0 \quad \text{all } a, b, c, d, e \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$f(a, b, c, d) = f(d, b, c, a) \quad \ldots \quad \ldots \quad (2)$$

Prove that

$$f(a, b, c, d) = -f(b, c, d, a).$$

3. Three dogs stand at the vertices of an equilateral triangle, whose sides are of length a. At the same instant, each begins to chase the one on his left, moving with uniform speed, v. How long does it take for the dogs to catch one another?

4. Dr. A. and Mr. B. held alternative lecture courses on M.W.F. at 9.0, both starting on Friday, January 14th, with non-empty classes.

Every Sunday, one of Mr. B's. students decides to change to Dr. A. Every Monday, half the students intending to attend each course oversleep, and skip the rest of the term.

Each holds his last lecture on Wednesday, March 9th. Mr. B. has an attendance of 5. What is the minimum number of students at Dr. A's. last lecture?

(Fractional students do not occur at any stage.)

5. Evaluate

$$\int_0^{\pi/2} \log(\sin\theta)\, d\theta$$

6. Using four 4's and any mathematical symbols, construct the numbers

$$7;\ 17;\ 37;\ 3{,}628{,}800.$$

7. A prime number of farmers each own a non-zero number of pigs, sheep, cows and horses, and no two farmers own the same number of any one animal. The total of pigs is equal to the square of the number of farmers. On those farms with an even number of pigs, the sheep number half the pigs, otherwise double the number. The total number of cows is the smallest possible.

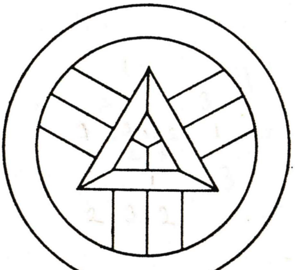
   If, when the number of sheep is the maximum possible, twice the number of sheep is 11 less than the minimum total of animals in this case, what is the smallest possible number of animals?

8. There are ten times as many seconds remaining in the hour as there are minutes remaining in the day. There are half as many minutes remaining in the day as there will be hours remaining in the week at the end of the day.

   What time is it on what day?

9. A pair of three-digit numbers having the hundreds and the units digits interchanged is called a pair of reversals, e.g. 724 and 427. A certain number is the sum of a pair of reversals, such that the smaller of the pair is the difference of a pair of reversals. Prove that this number is constant, and evaluate it.

10. Copy and colour the diagram with only 4 colours so that no two adjacent regions have the same colour. (Two regions are adjacent if and only if they have a common boundary.)



11. Give the next number of the following number sequences:—

(i) 2, 4, 8, 32 . . .
(ii) 3, 5, 11, 13, 17, 19, 29, . . .
(iii) 0, 1, 5, 23 . . .
(iv) 1, 1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 4, 1, 2, 5, 1, . . .
(v) 5, 11, 15, 16, 17, 18, 23, 25.

12. In *Ape and Essence,* we read of all children born with more than 12 fingers or fewer than 6, being slaughtered during Belial week. And since all available books were burnt off, we presume that the remaining children had to learn to count using their fingers and thumbs, giving rise to some confusion. Please illustrate this by performing some calculations on the assumption that you have (*a*) 7, and (*b*) 12, fingers altogether.

(i) Multiply 4632 by 121.

(ii) Find $\sum_{n=1}^{10} n$.

What would a 7-fingered person find $\sqrt{24442}$ to be, and a 12-fingered person $\sqrt{201}$?

(*Solutions on page* 21)

3D

THERE is nothing new under the sun. 3D dates, roughly speaking, from the time that eyes began to go around in pairs. One-eyed Cyclops, indeed, are mentioned in the classics; Odysseus was in the unenviable position of having rocks flung at his ship by one of them, Polyphemus by name. The fact that the latter did not score a hit is usually attributed to Odysseus' having already blinded his single eye. We cannot suppose, however, that the Cyclops would have met with any more success had he retained his sight.

For it is fairly common knowledge nowadays that a pair of eyes is indispensable to any accurate estimation of distance or depth. Naively, one could suppose that one way of judging how far away an object is is to make an estimate of its apparent size. This is true, up to a point; distance may be estimated in this way to an accuracy of an inch or two at arm's length. A simple demonstration of the scope and limitations of such an estimate is the following. A halfpenny is placed on a table, with about one-third of its diameter overhanging the edge; an attempt is made to flick the coin off with the fingertip—with one eye closed, of course. Consistent success is unlikely.

The fact that a similar experiment conducted with the use of both eyes is always (E & OE) successful shows the superiority of two eyes over one in the estimation of distance. It is this immediate awareness of depth which is responsible for the "solidity" of our surroundings.

In broad outline, the mechanism underlying stereoscopic vision is readily understood. The two eyes examine the scene before them from two different viewpoints, rather more than two inches apart. The effect of this is twofold: first, it is clear that the axes of the eyes will converge more in looking at a nearby object, than at a distant one. Second, the apparent position of a near object against a distant background will be different for the different viewpoints, and the nearer the object is the greater this difference will be. The first of these two effects seems to be the more important. A variation of the angle between the axes of the eyes gives rise to a muscular sensation which is normally interpreted in terms of the distance of the object observed. Of course, such a remark implies the existence of some sort of correlation between the positions of the two eyes; that such a correlation is indeed present in normal people is evident when one remembers that it is not usually possible to direct one eye arbitrarily while the other is kept stationary; discomfort is experienced when their axes converge to any great extent, and few people are able to make them diverge at all.

The fact that such muscular experiences are indeed interpreted in this way may be shown as follows. Some pattern of a repetitive nature is obtained (an expanse of wire netting, for example; at the moment, the author finds a typewriter keyboard convenient). It is possible to look at such an object in a number of ways, each making "sense" from the visual point of view. In the case of a typewriter, the keyboard may be looked at as any sensible person would look at it, viz., with the images of the QWERT etc. in the left eye coinciding with the same images in the right. Alternatively, it is possible to get these images in the left eye coinciding with the WERTY etc. in the right; in this case, the keys appear to be further away, and consequently larger, than one would expect. With the WERTY in the left coinciding with the QWERT in the right, the keys look smaller and nearer. The illusion can be due only to the changes in the angles of convergence of the eye axes, and there is no way in which such a change can be detected apart from muscular sensation. Presumably, the ability to catch a cricket ball falling from a cloudless sky depends to a great extent on this mechanism; comparison of the ball with its background cannot have anything to do with it—there is no background.

On the other hand, the presence of a background is sometimes important. A glimpse of a scene by a very brief flash of light, for example, will often give a distinct impression of depth, although there may be no time for the eyes to shift about. This means that the eyes are capable of using the difference in viewpoints in the second way mentioned above. Thus, though muscular sensation may well be the most important factor in stereoscopic vision, other factors—differences in images observed, apparent size, and possibly also the focussing mechanism of the eye—certainly have their place.

There are ways of improving the stereoscopic faculty. One is effectively to increase the distance between the viewpoints of the eyes; this is easily done with a few mirrors. Alternatively, a pair of telescopes may be used, one for each eye; differences due to differing viewpoints are more easily seen in the enlarged images. In the usual design of binoculars, both methods are employed; the objective glasses may often be more than twice as far apart as the eyepieces, giving a corresponding improvement in the stereoscopic faculty—over and above that provided by the magnification of the telescope systems. There are disadvantages, however; the view is invariably foreshortened by such an arrangement. Through binoculars, a cube examined face-on will appear as a square slab; looked at in any other way, it will assume a form which will be—for the purposes of this article—indescribable.

In order to make a photographic record of a scene capable of reproducing the impression of depth, two photographs will have to be taken, and thereafter viewed in such a way that the left and right eyes examine those taken from the left and right viewpoints respectively. The taking of such photographs presents no difficulty, nor does the viewing of them, as long as only one person is concerned. The main problem of 3D is to present a photographic record including the stereoscopic impression to each member of a large audience simultaneously. The only practicable way has been to project both pictures on to the same screen, with some means of distinguishing between them. The method now generally adopted is to polarise the light producing the pictures, one perpendicularly to the other; the screen is then viewed through the appropriate polarising filters; in this way each eye singles out its own image.

Stereoscopy has uses other than those of entertainment. The interpretation of aerial reconnaissance photographs is made much simpler by its use. Two photographs are taken vertically downwards, the viewpoints being a considerable distance apart. Examination of the stereo pair shows trees, chimneys, etc., standing out in pronounced relief. Even more celestial uses of stereoscopy have been known. At the turn of the century it provided a recognised technique for the discovery of asteroids. Any sky object moving over the background of stars in the interval between two photographs stood out clearly on stereoscopic examination. One asteroid discovered in this way was named *Stereoscopia.*

It is a far cry from the asteroids to 3D in Glorious Technicolor; it is the latter rather than the former that has given stereoscopy the wide familiarity and appeal that it now enjoys. But for this, we should be inclined to forget the remarkable ability conferred on us by our two eyes to interpret complicated sensations so clearly, rapidly and accurately in terms of depth. To be without this faculty would be a considerable handicap.

J. L. MARTIN.

Approximate Constructions for 7, 9, 11, 13-sided Polygons

GAUSS proved in 1796 that a straight edge and compass construction for a regular polygon having an odd number of sides is possible when, and only when, that number is either a prime Fermat number (that is, a prime of the form $2^{2^n} + 1$), or is the product of different Fermat numbers. Thus an accurate construction is impossible for 7, 9, 11 and 13-sided regular polygons. The four constructions given here for these polygons, though approximate, are easily proved to be accurate to within a few seconds of the required angles (namely, $2\pi/7$, $2\pi/9$, $2\pi/11$ and $2\pi/13$).

(i) Construct an angle $\cos^{-1} \frac{1}{10} (4 + \sqrt{5})$. If the angle is ABC, then by marking off arc lengths equal to AC on the circle AC centre B, we obtain a heptagon or seven-pointed star.

(ii) Construct an angle $\cos^{-1} \frac{1}{10} (5\sqrt{3} - 1)$ giving a 9-sided regular polygon.

(iii) Construct two angles $\cos^{-1} 8/9$, $\cos^{-1} 1/2$, and take their differences, whence the 11-sided regular polygon.

(iv) Construct two angles $\tan^{-1} 1$ and $\tan^{-1} (4 + \sqrt{5})/20$ and take their difference, giving 13-sided regular polygon.

J. C. OLDROYD.

Solutions to Problems Drive

(*See page* 15)

1. 0, 0. The Inca (1234567890) = our (5326741908).

2. $f(a, a, a, a) = 0$, by (1).
Also $f(c, a, a, a) + f(a, a, a, c) = 0$, so that by (2)
$$f(a, a, a, c) = 0.$$
Similarly we can show $f(b, c, d, d) = 0$, and
$$f(a, b, c, d) + f(b, c, d, a) + f(c, d, a, a) = 0,$$
hence the result.

3. $2a/3v$.

4. One.

5. $\frac{1}{2}\pi \log \frac{1}{2}$.

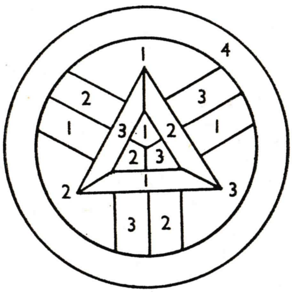
6. $4 + 4 - 4/4$; $4 \times 4 + 4/4$; $4! + 4/.4 + 4$; $(4 + 4 + 4/\sqrt{4})!$.

7. 495.

8. 10.48 p.m. on Sunday.

9. 1089.

10.



11. (i) 128 (2^p) or 256 ($\mu_p\, \mu_{p+1}$). (ii) 31 (pairs of primes). (iii) 119 ($n! - 1$). (iv) 4 (number of ways of factorising integers). (v) 27 (numbers whose squares have a 2 as digit).

12. (i) 630102, 538272. (ii) 66, 40; 143, 15.

Diophantus Smith—Solution

(*See page 7*)

Let us suppose that x men, y women, and z children finally benefited. We have

$$x + y + z = 100, \ 10x + \tfrac{1}{2}y + \tfrac{1}{8}z = 100 \qquad \ldots \qquad (1)$$

These give

$$79x + 3y = 700 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

Now we can express x and y in the forms

$$x = 3a + b, \ y = 79m + n,$$

where a, b, m and n are non-negative integers, and $b \leqslant 2$, $n \leqslant 78$. Also $700 = 3.79.2 + 226$.

Equation (2) becomes

$$79\,(3a + b) + 3\,(79m + n) = 3.79.2 + 226,$$

leading to

$$a + m = 2, \ 79b + 3n = 226. \qquad \ldots \qquad \ldots \qquad (3)$$

The second of these equations, along with the restrictions on b and n, determines b and n uniquely; in fact, $b = 1$, $n = 49$, by a little experiment.

The first of equations (3) permits $a = 0$, 1, or 2. Tabulating these possibilities, we have:

a	$x = 3a + 1$	$y = 79m + 49$	$z = 100 - n - y$
0	1	207	-108
1	4	128	-32
2	7	49	44

Until a meaning is attached to a negative number of beneficiaries, the last will remain the only solution. Seven men, 49 women, and 44 children therefore had cause to feel grateful for the memory of Diophantus Smith.

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Ever Been Had?—Solution

(See page 14.)

(i) John, where Willie had had 'had,' had had 'had had'; 'had had' had had full marks.

(ii) The above gives a sentence of the required type for $n = 0$. For $n = 1$ we might have:

In the punctuation of the above, A, where B had had ". . . had had 'had,' had had 'had had'; 'had had' had had . . .", had had ". . . had had 'had had,' had had 'had'; 'had had' had had . . ."; "had had had had had had had had had had had" had had two possible interpretations.

The string of *had*'s consists of the three sets of eleven in double quotes, with six additional ones:

$u_1 = 3u_0 + 6$ (where u_n = number of *had*'s in the nth sentence).

By a similar dodge, the fact that the sentence for $n = 1$ is punctuable in two different ways allows us to construct a sentence for $n = 2$. As before:

$$u_2 = 3u_1 + 6.$$

Continuing thus, we have generally

$$u_{n+1} = 3u_n + 6$$

with $u_0 = 11$. The solution of this recurrence relation is left as an exercise for the student.

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Book Reviews

The Fairy Chess Review. Ed. D. NIXON. 1s. 8d. per copy; 10s. per year.

This bimonthly journal will interest devotees of the unorthodox forms of chess, and contains a large number of problems in grid chess, helpmates, selfmates, series mate, etc. It is obtainable from the Editor, 49, Manor Street, Middlesborough, Yorks. G. T.

Factorial Analysis for Non-Mathematicians. C. J. ADCOCK. (Melbourne University Press.) 17s. 6d.

It may seem an impertinence to review in *Eureka* a book intended for non-mathematicians. However, statistics is a branch of the subject with which the average Cambridge mathematician makes little contact.

This book is intended to teach psychologists and others the elements of factorial analysis, that is to say, the procedure of statistically eliciting the factors producing their experimental results. Any book that tries to explain a mathematical procedure to non-mathematicians tends to be nothing more than a recipe book. Dr. Adcock has not been able to avoid this difficulty. The starts of methods are given without their completion: for instance, tetrachoric correlation is described without giving calculations for ρ. The student of this book will know what to do but not why to do it. This leads sometimes to difficulties outside textbook examples, and may lead to it being said that

"Though they wrote it all by rote,
They did not write it right."

One must also comment that 17s. 6d. is a large price for a small volume (88 pp). R. I. P.

Harmonic Analysis and the Theory of Probability. S. BOCHNER. (University of California Press.) 35s.

In the last 30 years, the theory of probability has stimulated and been stimulated by the rapid developments in harmonic analysis. This volume shows clearly the effect of this continual interchange upon the growth of both subjects.

The Fourier transform, the characteristic function in this context, has long been an effective instrument of research in probability. In the early chapters of this book, the principal features are given of the theory of Fourier series and Fourier-Stieltjes integrals, with their principal closure properties and their application to the Laplace and heat equations, the theory of completely monotone functions in one and several variables and of Mellin and Laplace transforms.

The last two chapters are of particular value. We are introduced to the idea of characteristic functionals, or Fourier transforms in a Euclidean-like space of infinitely many dimensions and their application to the analysis of stochastic processes. Some of this work has been published by Bochner in various journals, and here a coherent and connected account is given. This new tool promises to be very powerful, and we may well see many further developments in its use in the next few years. This part of the book alone makes it worth the money.
W. P. B.

The Real Projective Plane. H. S. M. COXETER. (Cambridge University Press.) 27s. 6d.

The economy and elegance of the axiomatic development of projective geometry is always a delight, and a well produced book on this subject should not disappoint. Starting from the axioms of incidence, order and continuity, the reader is carried over a well charted course across two-dimensional projectivities and conics to the beginnings of affine geometry. Analytical geometry is mentioned in the last chapter to show that the synthetic geometry developed has the same properties as the plane of real homogeneous co-ordinates.

The book is written in a style which is lively and attractive, if rather prolix. Many of the chapters commence with sections that outline the historical development of the ideas to be discussed, and provide a coherent background to the many names associated with various parts of the theory. Too rarely is this done in mathematical textbooks and these names are frequently meaningless labels. On the other hand, the wordiness of the proofs may irritate an advanced reader without making the journey very much easier for the student.

Nevertheless, a pleasing book.

M. W. P.

Biomathematics. CEDRIC A. B. SMITH. (Charles Griffin & Co., Ltd.) 80s.

The first edition of *Biomathematics* was published in 1923 under the authorship of W. M. Feldman, and the second in 1935, just twenty years ago. The application of mathematics to biological problems was then novel and the prospects of further development exciting. Much has been done in these twenty years, and mathematics has become a tool of increasing importance to the biologist. Dr. Smith, no stranger to readers of *Eureka*, has completely revised the earlier book, and in rewriting it, has included a considerable amount of material additional to that in the previous editions.

This book seeks to explain the principles and techniques of mathematics to students of the natural sciences. This aim is an ambitious one and the scope of the book is necessarily wide. Its 700-odd pages include chapters on elementary algebra and trigonometry, calculus, series, vectors, equation-solving techniques, matrices, probability and simple statistical theory. There is also an interesting note on the Colson arithmetical notation. No prior mathematical knowledge is required of the student, only the desire to acquire some. This is no book for the formula-plug; it stresses the principles involved and develops the essentials of the argument in a manner readily comprehensible to the neophyte.

Dr. Smith is a stylist. In this book we come across whimsical interludes that cannot but charm; the reader goes on refreshed. His writings in a lighter mood have been received with delight and Herlock Soames, the Sacred Jewel of the Ngboglus and the others (all soberly indexed) will also give much pleasure.

Professional mathematicians also would do well to examine *Biomathematics*. The extent to which simple mathematical techniques (other than statistical ones) can be applied to biological problems may come as a surprise. Many of the treatments, designed as they are to reveal simply the salient ideas of the mathematics, will also be worthy of attention.

O. M. P.

The Sources of Eddington's Philosophy. HERBERT DINGLE. (Cambridge University Press.) 3s. 6d.

This book is the text of the eighth Arthur Stanley Eddington Memorial Lecture; it is an attempt by one who knew Eddington well to account for his distinctive outlook on the universe. The main mistake made by Eddington, it seems, was living at the time he did; had he lived a decade or two later, his contribution to physical philosophy might have been quite different. As it was, however, he had plenty of time to become well versed in the outlook of Victorian physics before the advent of Relativity and other new developments; these came too late to allow him fundamentally to change this outlook, which he had largely adopted as his own.

Thus he postulated an "external" world, a rather vague entity lying behind the physical world, certain of whose "conditions" the physical world purported to describe. The coming of Relativity, with its attendant abolition of certain absolutes—absolute time, absolute rest, etc.—ought to have provided a warning that the need for such an external world might no longer exist; whatever its uses in any other philosophy, it might be totally unnecessary and undesirable in a physical philosophy.

The author, however, points out that a mistaken or antiquated philosophy by no means debars one from making a positive contribution to physics. A distinction is therefore drawn between Eddington's philosophy and his physics (although Eddington himself made none), and the suggestion is made that if Eddington's theoretical work were to be liberated from his "unspeakable" philosophy, the result might well be of great value. This has not yet been fully done.

Such in outline is the content of the book. It makes interesting reading while presenting nothing startlingly controversial, and appears to accomplish what it sets out to do—to account for Eddington's philosophical outlook in terms of his philosophical environment. No doubt Eddington, had he the opportunity, would have questioned some of Professor Dingle's remarks, and maintained that he held his views independently of the Victorian physicists—but then what one of us likes to admit that our outlook is not entirely of our own making?

J. L. M.

Algebraic Geometry. C. V. Durell. (G. Bell & Son, Ltd.) 18s. 6d.

The purpose of this book is best summed up by quoting from its preface—namely "to prepare the reader for the reception of the abstract ideas of geometry which he will meet at the University." In the introduction the essential differences between Spatial Geometry and Abstract Geometry are carefully pointed out and in the first few chapters the new foundations are laid. The principle of duality is introduced early and is employed advantageously to drive home the new abstract ideas by means of well-balanced repetition.

The author expresses a hope that mathematical specialists will be able to read this book without makking much demand on the teachers. This is perhaps a little optimistic for the initial stages, but the subject is developed logically and clearly, so that to anyone who has grasped the underlying principles, the later applications should present few problems.

The greater part of the book is concerned with plane projective algebraic geometry, which is developed far enough for invariants and the harmonic envelope of two conics to be touched on. The last two chapters deal excellently with the relation between projective and cartesian geometry, and there is also one chapter in which solid geometry is considered. This latter is perhaps a little confusing in that the six members of the ratio-set defining a line are given in their usual form —that is, with the suffix notation, which differs from that used in the rest of the book. The co-existence of these two notations is not conducive to clarity.

One of the difficulties of geometric examples is to know which is the best way to approach them. Is the solution best expressed in algebraic or geometric language? It is therefore a welcome quality of this book that there is throughout a comparison of methods. Many examples are worked out in two—sometimes three, different ways and one is able readily to assess their relative merits. Each chapter is supplied with three or four sets of exercises; some are straightforward, others it is suggested should be left to a second reading.

This book was written primarily for those taking entrance scholarships to the universities; such though is its scope and quality, that it would be a pity if it were to be restricted to this field alone. B. R.

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