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EUREKA

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EUREKA

Editor: *P. M. Lee (Churchill)*

Jubilee Issue

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THE JOURNAL OF THE ARCHIMEDEANS
The Cambridge University Mathematical Society; Junior Branch of the Mathematical Association

Contents

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Editorial

TWENTY-FIVE years ago, *Eureka* was first published. At the time, the aim of the magazine was summarised by the then President of the Archimedeans in the following words:—

". . . The chief thing is to make it interesting to every Cambridge mathematician, to help build up the corporate interest in the subject . . . to link together students, researchers and dons, other English and foreign universities. We must aim to stimulate informed discussion, especially as to Cambridge questions. . . ."

In the humble belief that our predecessors have in some measure achieved these ambitions, we offer this jubilee issue of *Eureka*, most of which is devoted to reprinting a few of those articles which we have enjoyed reading in previous issues and which we feel deserve to be known to a new generation. Next year, *Eureka* will return to its accustomed form (all contributions, from Cambridge or outside, being welcome), but this year we salute the distinguished company who have written, edited and printed the last twenty-six *Eureka*s.

Greetings

I HAVE great pleasure in congratulating *Eureka* on the occasion of its attaining the age of 25 years. This is quite an advanced age for an undergraduate (or mainly undergraduate) journal, but I am glad to say that I see no signs of senility.

For 25 years *Eureka* has provided an opportunity for some of our most alert and penetrating minds to publish their ideas, and many of their contributions are of permanent interest and have found a place in mathematical literature.

The Archimedeans have a wide variety of activities; some of them may die out in the course of time, as tastes change, but I hope that the Society will always regard the publication of *Eureka* as one of its most important functions.

H. DAVENPORT,
Chairman of the Faculty of Mathematics.

ON BEHALF of the first Editorial Committee, I have pleasure in greeting *Eureka* on its twenty-fifth birthday.

Our first Editorial expressed our hopes and ideals for the journal and it is good to see these being fulfilled. It is a matter of personal satisfaction to see that the name, the cover, and the format have withstood the test of time and remained unchanged since No. 1. In that issue, we acknowledged the sound advice and good workmanship of our printers. Their services have been retained by our successors, all of whom would doubtless like to endorse this repetition of our appreciation of their help and skill.

Eureka has grown up. In its maturity, may it continue to serve a use to those young mathematicians without whose support it cannot flourish.

Sheffield.	Arthur Jackson.
Thurso.	E. P. Hicks.
Keyport, New Jersey.	Jessie MacWilliams (*née* Collinson).

The Archimedeans

The following anthem, which was written for the society by Mr. W. Hope-Jones in 1938, *seems to have fallen out of use:*

All praise to Archimedes,
  Who weighed the royal hat,
Displacing quarts of h. and c.,
  Upon the bathroom mat.
For that unending decimal,
  We mortals know as π,
He found that three-and-one-seventh,
  Was just a bit too high.

All praise to Arthur Eddington,
  Who proved I don't know what,
Except that ev'rything you think's
  Exactly what it's not.
He knows what Albert Einstein's
  Equations are about;
And that's where he has you and me,
  And Einstein up the spout.

It may be sung to the tune of Hymn 341, *A. and M. Descending to the realm of everyday life, we now print the Secretary's Report.*

The Society has once again had a very successful year with the evening meetings, on the whole, very well attended. Notable successes were those of Professor Scott, who demonstrated the difference between the torus and the sphere with the aid of his pyjamas, and of Professor Bondi whose talk on "Gravitation" attracted an audience from many other faculties. The tea-talks also drew large numbers; Professor Thurston Dart gave an excellent lecture on "Composers and Computers," whilst Professor Besicovitch and his own peculiar card game resulted in the Secretary and several friends wasting many a pleasant hour. At the start of the year the programme card showed a fortnightly computer group, a puzzles and games ring, a music group, a bridge group, a mathematical models group and a play-reading group. The end of the year, unfortunately, saw these last two enter the defunct category, although it is hoped that they will not remain in this state. The others have, however, flourished; the computer group now meets once a week, musicians are catered for by both the music group and the newly formed chamber music group, and the bridge players are forced to play with even more worn cards.

To start this year's evening meetings Professor V. C. A. Ferraro will give a talk with lantern slides on the subject of the "Magnetosphere." This is followed by the return to Cambridge of Professor E. C. Zeeman with a lecture on "Lens Spaces," Professor H. G. Eggleston on the "Kakya Problem," and to finish the Michaelmas Term, a talk by Professor D. R. Cox on "Some Industrial Applications of Probability." Between these last two talks we have the annual careers meeting, with a schoolmaster, an actuary, the Deputy Head of the Operations Research Branch of the National Coal Board and a Principal Scientific Officer from Farnborough each giving us a little insight into his own chosen trade or profession. The Lent Term is opened by Professor D. Bohm on the subject of "Space, Time and the Quantum Theory," and in fairly close succession we have the opportunity of hearing Professor W. H. Cockcroft, Professor C. A. Coulson on "Vibrations of Large Systems," and Professor W. K. Hayman, whose talk entitled "Circles, Spheres and Condensers" is on simple applications of symmetrisation to problems in applied mathematics. The tea meetings are again presided over by research students and the subjects range from Spiral Galaxies to an intriguing East African game called Mweso. The Problems Drive will again be held in February and we hope it will receive as much support as it has done in the past. There will be visits to such places as the Mathematics Division of the National Physics Laboratory at Teddington, as well as the usual excursions to various London threatres, including, if possible, a Gilbert and Sullivan opera in the Lent Term. It is hoped that there will be sufficient support to hold a dinner at Christmas, and also to launch a reasonable number of punts at the annual punt party in May.

The programme has been designed to cater for the needs of all members of the Society, but the Secretary would appreciate any suggestions as to possible improvements or alterations, which may be made either directly or through the book kept for this purpose in the Arts School.

M. A. MILLER, *Secretary.*

Problems Drive 1964

SET BY M. S. PATERSON AND MISS J. E. HEARNSHAW

A. Write down as many primes as you can which are palindromic (read the same in both directions) in binary notation, e.g. 17 (= 10001).

B. At noon precisely, a train leaves A for B, and another leaves B for A. They pass after 51 minutes. Each train stays 27 minutes at its destination and then returns by the same route. The trains from A and B travel throughout with constant speeds of 23 m.p.h. and 39 m.p.h., respectively. At what time do they pass for the second time?

C. (*a*) 3, 4, 6, 8, 12, 14, . . . add two terms.

(*b*) 61, 52, 63, 94, 46, . . . add one term.

(*c*) 1, 2, 3, 5, 16, 231, . . . add one term.

(*d*) 65, 58, 72, 107, . . . add one term.

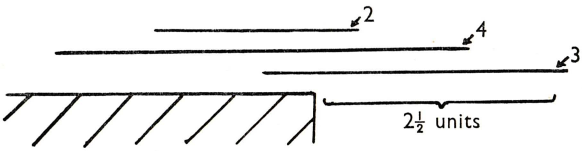
D. Your are given a 15-pint, a 10-pint and a 6-pint measure, and an unlimited water supply. Find a method of obtaining exactly 1 pint in each of two measures, using the fewest possible number of operations. (Marking of measures is not permitted. An operation is either filling or emptying one measure, or transferring water from one to another.)

E. In the multiplication sum

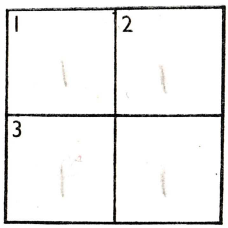
$$\begin{array}{r} \mathrm{A\ B\ C\ D} \\ \mathrm{E} \\ \hline \mathrm{F\ G\ H\ I\ J} \\ \hline \end{array}$$

the letters represent different digits in the scale of ten. If E = 4, what is (A + B + C + D)?

F. There are three uniform thin planks of lengths (and weights) 4, 3 and 2 units, respectively. In this diagram, they are arranged on the edge of a shelf so as to project a distance of $2\frac{1}{2}$ units over the edge. How can they be rearranged to project the maximum possible distance?



G. Solve the following cross-number puzzle (in integers in the scale of ten).



ACROSS:	DOWN:
1. a	1. c
3. b	2. d

where c is prime and $a = 2c + d - 2b$, $b = (a^2 + c^2)/2d$, and $d = (a - b)^2 + c$.

H. My friend tosses two coins and covers them with his hand. "Is there at least one 'tail'?" I ask. He affirms this (a).

Just then he accidentally knocks one of them to the floor (b). On finding the dropped coin under the table, we discover it to be a 'tail' (c).

"That is all right," he says, "because it was a 'tail' to start with" (d).

At each point (a), (b), (c) and (d) of this episode I calculated what, to the best of my knowledge, was the probability that both coins showed 'tails' at the time. What were these probabilities?

I. Find the last digit of:—

$$33333^{77777} - 77777^{33333}$$

J. The planet Kophikkup is in the shape of a torus or ring-doughnut. There is a direct mono-rail line from each of the four space-ports to each of the major cities. No lines join or cross. What is the greatest possible number of major cities? Draw a diagram for this case.

K. I have a wire model consisting of the edges of a cube. If I remove exactly three of the edges, how many different structures can I produce?

L. When my sister is four times as old as she was when I was twice as old as my brother, my brother will be two-thirds as old as I will be. My brother and I are teenagers; how old was my sister on her last birthday?

Mathematical Association

President: Miss I. W. Busbridge, M.A., D.Phil., D.Sc.

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object but at bringing within its purview all branches of elementary mathematics.

The subscription to the Association is 21*s.* per annum: to encourage students and those who have recently completed their training the rules of the Association provide for junior membership for two years at an annual subscription of 10*s.* 6*d.* Full particulars can be had from The Mathematical Association, Gordon House, Gordon Square, London, W.C.1.

The *Mathematical Gazette* is the journal of the Association. It is published four times a year and deals with mathematical topics of general interest.

■ ■ ■

"What shall we say of the mathematics? Shall we deem them to be the delirious ravings of madmen? Nay; we cannot read the writings of the ancients on these subjects without the highest admiration."

John Calvin, *Institute* and *Eureka*, **17**.

Mathematics and Games

BY P. M. GRUNDY

IN ANY game between two opponents playing alternately, in which the legality of moves is not governed by chance, it is intuitively clear that there must be some best method of playing; and if both play correctly the result of the game is determined by the initial position. In fact if, when it is A's turn he cannot force a win then, however A moves, B can follow with a move after which A again cannot force a win. By repeating these tactics B can at least secure a draw, which should be the conclusion of the game unless B can force a win. This reasoning holds even for card games, with the snag that there the players are not allowed to know the state of the game.

There is a remarkably simple theory for the game with piles of matches (or nuts, etc.) called Nim. The move consists of taking any number ($\geqslant 1$) of matches from any one heap, and the player taking the last match wins.* The theory depends on the operation of Nim Addition, denoted by $\underset{N}{+}$: when the numbers $x_1, x_2 \,..$ on the heaps are expressed in the scale of 2 as $x_i = \sum_j a_{ij} 2^j$ ($a_{ij} = 0$ or 1; $i = 1, 2 \,..$) and $b_j = 0$ or 1 according as $\sum_i a_{ij}$ is even or odd, the Nim Sum $x_1 \underset{N}{+} x_2 \underset{N}{+} \,..$ is defined to be $\sum_j b_j 2^j$. This operation obeys, like ordinary addition, the commutative and associative laws for bracket-moving; and also has the fundamental property $x \underset{N}{+} x = 0$ (any x). Now call the position $(x_1, x_2 \,..)$ winning (W) if $x_1 \underset{N}{+} x_2 \underset{N}{+} \,.. = 0$, and losing (L) if not. Then the terminal state

0	1	2	3	4	5	6	7	..
1	0	3	2	5	4	7	6	..
2	3	0	1	6	7	4	5	..
3	2	1	0	7	6	5	4	..
4	5	6	7	0	1	2	3	..
5	4	7	6	1	0	3	2	..
6	7	4	5	2	3	0	1	..
7	6	5	4	3	2	1	0	..
..	..	..	..	..	..	..	..	..

Nim Addition Table
e.g. $3 \underset{N}{+} 6 = 5$

* There is a variation in which the last mover loses.

Cf. C. L. Bouton, *Annals of Mathematics* (Harvard), 2nd Series, III (1901–2); W. W. R. Ball, *Mathematical Recreations and Essays,* Ch. I; Hardy and Wright, *Theory of Numbers*, etc.

(all $x_i = 0$, when the games finishes) is W; and also this classification of states obeys the characteristic recurrence-relations :—

$$(1) \quad \begin{cases} \text{from a non-terminal L it is always possible to move} \\ \quad \text{to a W,} \\ \text{from a W it is impossible to move to a W.} \end{cases}$$

A player who once moves to a winning position can evidently continue to do so at his subsequent moves, and so eventually win.

More generally, if a draw is impossible, a similar classification exists, provided the moves for the two players are identical and depend only on the state of the game. First let each terminal state be classified W or L according as (by the rules of the game) the player moving to it wins or loses. Owing to the fact that states of the game form a "partially ordered set" it is now possible, with these end-conditions fixed, to work backwards as far as required using (1) at each stage. This procedure is quite practicable, though laborious unless a lucky guess can be made. It follows again from (1) that a player who once moves to a W state can continue to do so, and eventually win either by himself moving to a terminal W, or by his opponent moving to a terminal L.

In Nim a state $X = (x_1, x_2 \ldots)$ may be regarded as reducible to a "product" $X_1 X_2 \ldots$ (in any order) of irreducible states $X_1 = (x_1, 0 \ldots)$, $X_2 = (x_2, 0 \ldots) \ldots$; and at each move just one of the irreducible components (factors) is changed (perhaps into a reducible factor in other similar games). For games of this restricted type, in which the last mover wins, the work of classification can be made easier without recourse to guessing by means of a function $\Omega(X)$ ($=$ integer $\geqslant 0$) of the variable X ($=$ general state of the game) determined by the properties :—

$$(2) \quad \begin{cases} \text{any single move alters the value of } \Omega(X), \\ \text{if } 0 \leqslant \omega < \Omega(X), \text{ the value of } \Omega \text{ can be decreased} \\ \quad \text{to } \omega \text{ in one move,} \\ \Omega(X) = 0 \text{ when X is terminal.} \end{cases}$$

Taking the existence of Ω for granted, comparison of (2) with (1) shows that X is a W-state if and only if $\Omega(X) = 0$. It may also be deduced from (2) that

$$(3) \qquad \Omega(XY\ldots) = \Omega(X) \underset{N}{+} \Omega(Y) \underset{N}{+} \ldots$$

This accounts for the advantage of working with Ω, since *all the winning positions will be known if the value of Ω is calculated merely for irreducible states.* In practice the values of Ω may be found by working back from the end of the game, using (2) and (3).

The following game with piles of matches is an example:— The move consists of taking any pile and dividing it into two *unequal* parts. The game is one in which the last mover wins; and its states may be described by sets of co-ordinates x_1, x_2 .., giving the numbers in the heaps. We may use the same law of combination $(x_1, x_2, \ldots).(y_1, y_2, \ldots) = (x_1, x_2, \ldots y_1, y_2, \ldots)$, obtained simply by putting the two sets of heaps side by side, as before; and then, since the game is of the type just considered, we have as a special case of (3):

$$(4) \qquad \Omega\,[(x_1, x_2, \ldots)] = \Omega\,[(x_1)] \underset{\mathrm{N}}{+} \Omega\,[(x_2)] \underset{\mathrm{N}}{+} \ldots$$

The irreducible states are those (x) with only one co-ordinate (i.e. one heap of x matches) and the law of moving is that (x) may be replaced by (u, v) if $\begin{cases} x = u + v,\ u \neq v \\ \text{all co-ordinates} \geqslant 1. \end{cases}$

The values of $\Omega\,[(x)]$, obtained (with the help of (4)) according to (2) with this law are

$$\begin{array}{rl} x = & 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17, \ldots \\ \Omega\,[(x)] = & 0,\ 0,\ 1,\ 0,\ 2,\ 1,\ 0,\ 2,\ 1,\ \ 0,\ \ 2,\ \ 1,\ \ 3,\ \ 2,\ \ 1,\ \ 3,\ \ 2, \ldots \end{array}$$

Owing to the curious repetitions, these values are quite easy to remember. Thus, as in Nim, a player using the theory can almost invariably win without visible calculation.

Eureka, **2.**

Postal Subscriptions and Back Numbers

FOR the benefit of persons not resident in Cambridge we have a postal subscription service. You may enrol as a personal subscriber and either pay for each issue on receipt or, by advancing 10*s*. or more, receive future issues as published at approximately 25 per cent. discount, with notification when credit has expired. The rates this year are:—

For those who pay on receipt 2*s*. 6*d*. (2½*d*. *postage*).

For those who pay in advance 2*s*. (*post free*).

Some copies of *Eureka* Nos. 19, 22 (2s. each) and 24, 25, 26 (2s. 6d. each)—are still available (postage 2½d. extra on each copy). Set of five, 9s. 6d. post free. We would be glad to buy back any old copies of Nos. 1 to 10, 12 and 14 which are no longer required.

Cheques and postal orders should be made payable to "The Business Manager, *Eureka*," and addressed to The Arts School, Bene't Street, Cambridge, England.

Mathematics in Wartime

By Professor G. H. Hardy

The editor asked me at the beginning of term to write an article for *Eureka*, and I felt that I ought to accept the invitation; but all the subjects which he suggested seemed to me at the time quite impossible. "My views about the Tripos"—I have never really been much interested in the Tripos since I was an undergraduate, and I am less interested in it now than ever before. "My reminiscences of Cambridge"—surely I have not yet come to that. Or, as he put it, "something more topical, something about mathematics and the war"—and that seemed to me the most impossible subject of all. I seemed to have nothing at all to say about the functions of mathematics in war, except that they filled me with intellectual contempt and moral disgust.

I have changed my mind on second thoughts, and I select the subject which seemed to me originally the worst. Mathematics, even my sort of mathematics, has its "uses" in war-time, and I suppose that I ought to have something to say about them; and if my opinions are incoherent or controversial, then perhaps so much the better, since other mathematicians may be led to reply.

I had better say at once that by "mathematics" I mean *real* mathematics, the mathematics of Fermat and Euler and Gauss and Abel, and not the stuff which passes for mathematics in an engineering laboratory. I am not thinking only of "pure" mathematics (though that is naturally my first concern); I count Maxwell and Einstein and Eddington and Dirac among "real" mathematicians. I am including the whole body of mathematical knowledge which has permanent aesthetic value, as for example, the best Greek mathematics has, the mathematics which is eternal because the best of it may, like the best literature, continue to cause intense emotional satisfaction to thousands of people after thousands of years. But I am not concerned with ballistics or aerodynamics, or any of the other mathematics which has been specially devised for war. That (whatever one may think of its purposes) is repulsively ugly and intolerably dull; even Littlewood could not make ballistics respectable, and if he could not, who can?

Let us try then for a moment to dismiss these sinister by-products of mathematics and to fix our attention on the real thing. We have to consider whether real mathematics serves any purposes of importance in war, and whether any purposes which it serves are good or bad. Ought we to be glad or sorry, proud or ashamed, in war-time, that we are mathematicians?

It is plain at any rate that the real mathematics (apart from the elements) has no *direct* utility in war. No one has yet found any war-like purpose to be served by the theory of numbers or relativity or quantum mechanics, and it seems very unlikely that anybody will do so for many years. And of that I am glad, but in saying so I may possibly encourage a misconception.

It is sometimes suggested that pure mathematicians glory in the "uselessness" of their subject, and make it a boast that it has no "practical" applications.* The imputation is usually based on an incautious saying attributed to Gauss,† which has always seemed to me to have been rather crudely misinterpreted. If the theory of numbers could be employed for any practical and honourable purpose, if it could be turned directly to the furtherance of human happiness or the relief of human suffering (as for example physiology and even chemistry can), then surely neither Gauss nor any other mathematician would have been so foolish as to decry or regret such applications. But if on the other hand the applications of science have made, on the whole, at least as much for evil as for good—and this is a view which must always be taken seriously, and most of all in time of war—then both Gauss and lesser mathematicians are justified in rejoicing that there is one science at any rate whose very remoteness from ordinary human activities should keep it gentle and clean.

It would be pleasant to think that this was the end of the matter, but we cannot get away from the mathematics of the workshops so easily. Indirectly, we are responsible for its existence. The gunnery experts and aeroplane designers could not do their job without quite a lot of mathematical training, and the best mathematical training is training in real mathematics. In this indirect way even the best mathematicians becomes important in war-time, and mathematics are wanted for all sorts of purposes. Most of these purposes are ignoble and dreary—what could be more soul-destroying than the numerical solution of differential equations? —but the men chosen for them must be mathematicians and not

* I have been accused of taking this view myself. I once stated in a lecture, which was afterwards printed, that "a science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life"; and this sentence, written in 1915, was quoted in the *Observer* in 1939. It was, of course, a conscious rhetorical fluorish (though one perhaps excusable at the time when it was written).

† To the effect that, if mathematics is the queen of the sciences, then the theory of numbers is, because of its supreme "uselessness," the queen of mathematics. I cannot find an accurate quotation.

laboratory hacks, if only because they are better trained and have the better brains. So mathematics is going to be really important now, whether we like it or regret it; and it is not so obvious as it might seem at first even that we ought to regret it, since that depends upon our general view of the effect of science on war.

There are two sharply contrasted views about modern "scientific" war. The first and the most obvious is that the effect of science on war is merely to magnify its horror, both by increasing the sufferings of the minority who have to fight and by extending them to other classes. This is the orthodox view, and it is plain that, if this view is just, then the only possible defence lies in the necessity for retaliation. But there is a very different view which is also quite tenable. It can be maintained that modern warfare is *less* horrible than the warfare of pre-scientific times, so far at any rate as combatants are concerned; that bombs are probably more merciful than bayonets; that lachrymatory gas and mustard-gas are perhaps the most humane weapons yet devised by military science, and that the "orthodox" view rests solely on loose-thinking sentimentalism. This is the case presented with so much force by Haldane in *Callinicus*.* It may also be urged that the equalisation of risks which science was expected to bring would be in the long run salutary; that a civilian's life is not worth more than a soldier's, or a woman's than a man's; that anything is better than the concentration of savagery on one particular class; and that, in short, the sooner war comes "all out" the better. And if this be the right view, then scientists in general and mathematicians in particular may have a little less cause to be ashamed of their profession.

It is very difficult to strike a balance between these extreme opinions, and I will not try to do so. I will end by putting to myself, as I think every mathematician ought to, what is perhaps an easier question. Are there *any* senses in which we can say, with any real confidence, that mathematics "does good" in war? I think I can see two (though I cannot pretend that I extract a great deal of comfort from them).

In the first place it is very probable that mathematics will save the lives of a certain number of young mathematicians, since their technical skill will be applied to "useful" purposes and will keep them from the front. "Conservation of ability" is one of the official slogans; "ability" means, in practice, mathematical, physical, or chemical ability; and if a few mathematicians are "conserved" then that is at any rate something gained. It may be a bit hard on the classics and historians and philosophers, whose

* J. B. S. Haldane, *Callinicus; a defence of chemical warfare* (Kegan Paul, 1924).

chances of death are that little much increased; but nobody is going to worry about the "humanities" now. It is better that some should be saved, even if they are not necessarily the most worthy.

Secondly, an older man may (if he not *too* old) find in mathematics an incomparable anodyne. For mathematics is, of all the arts and sciences, the most austere and the most remote, and a mathematician should be of all men the one who can most easily take refuge where, as Bertrand Russell says, "one at least of our nobler impulses can best escape from the dreary exile of the actual world." But he must not be too old—it is a pity that it should be necessary to make this very serious reservation. Mathematics is not a contemplative but a creative subject; no one can draw much consolation from it when he has lost the power or the desire to create; and that is apt to happen to a mathematician rather soon. It is a pity, but in that case he does not matter a great deal anyhow, and it would be silly to bother about him.

Eureka, **3**.

What is a Mathematician?

By Harold Jeffreys, M.A., D.Sc., F.R.S.

Professor Hardy has defined mathematics partly by enumeration of shining examples of mathematicians, partly by reference to the whole body of mathetmatical knowledge with permanent aesthetic value. The criteria do not seem to be very clear. Does the permanent value depend on the results or the method? Many of the most important results of pure mathematics are now stated in ways that their original authors would find it difficult to understand, and the original proofs would often fail to pass muster by modern standards, being either non-rigorous or hopelessly long-winded. What, for instance, would Fourier make of his theorem as presented by Titchmarsh, and what would happen to a Tripos candidate who gave Fourier's original proof? Newton's *Principia* is a monumental work, but how much of it survives in anything like its original form?

It seems to me that "mathematician" is used in several different senses. To most people a mathematician is anybody that can solve a quadratic equation; in Cambridge he is perhaps anybody that has taken, or proposes to take, the Mathematical Tripos, though it becomes doubtful whether he retains the title if he takes any other Tripos too. But many people make use of mathematics and yet do not consider themselves mathematicians. To such people mathematics is a tool, and not a direct source of interest. Professor Dirac, for instance, considers himself a physicist, and says quite explicitly that he considers mathematics a tool. His work, also, makes far more appeal to experimental physicists than it does to most pure mathematicians. In spite of Professor Hardy's admission of him as a mathematician, Dirac has more in common with Rutherford than with Hardy. Now I think that this is a genuine distinction. If people must be classified as specialists, it seems to me much more important that they should be classified with regard to what they talk about than with regard to the technique they use. Dirac should be called a theoretical physicist, not an applied mathematician. The latter term would then be free to denote those who are interested in physical problems only as illustrations of mathematical methods. I mention no names, considering the description abusive.

Unfortunately the Mathematical Tripos cannot get on without applied mathematics. Theoretical physicists, whether they hope ever to make original advances of their own, or whether they would be satisfied to be able to take an intelligent interest in the work of others, must learn a certain amount of pure mathematics and the general principles and methods of theoretical physics. But problems of intrinsic physical interest usually turn out to be either bookwork or too hard to be done in the time available in an examination! The result is that ability to apply the principles can be tested in the examination only by reference to experiments invented for the purpose: rolling illustrated by a sphere inside a vertical cylinder, with no slipping at all at the point of contact, potential problems for boundaries that no experimenter could construct, and so on. That is the trouble about Parts I and II: either the questions apparently on theoretical physics will be too hard, or they will be artificial and unsatisfactory, thereby becoming applied mathematics. In Part III the difficulty is less serious, since a question that takes three hours to answer, if the candidate knows how, is possible.

In dynamics there is a special difficulty. All the technique needed for the enormous majority of actual problems was known to Laplace, with the possible exception of the modified Lagrangian function, due to Routh. For these problems these are still the best methods. The reason why dynamics to this stage sometimes appears stagnant is that it was the first branch of theoretical physics to be developed, and when Newton, d'Alembert, Lagrange, Euler and Laplace had all played their parts in improving the methods it is hardly surprising that little further remained to be done. Nevertheless gyroscopic motion and small oscillations remain interesting (though the associated mathematics is now called algebra). The more advanced development beginning with Hamilton's equations and the Hamilton-Jacobi theorem makes the harder problems easier, but it makes the easier ones harder.

De gustibus. . . . But I must dissent from Professor Hardy's remark that ballistics and aerodynamics are ugly and dull. It *is* interesting that for bodies of some shapes moving through a fluid the force is nearly perpendicular to the direction of motion; there *is* beauty in the circulation theorem of aerofoil lift and in Prandtl's theory of induced drag; also in G. I. Taylor's treatment of various problems of bodies moving at high velocity in compressible fluids. Incidentally the circulation theorem also explains why people catch crabs in rowing; and the ballistics problem is substantially the same as one that arises in the evolution of the solar system, a matter of interest though hardly of economic or military importance. The devil's possession of the good tunes need not be undisputed. Again, some of us enjoy the numerical solution of differential equations. I do myself in moderation; but where should we be without those, from Briggs and Napier to the British Association Tables Committee, who find their chief joy in numerical computation? E. W. Brown could write down the equations of the moon's motion in two minutes; it took him thirty years to solve them to the accuracy needed for comparison with observation. But when I met him his soul always seemed to be doing very well. And that is the ultimate difference between the mathematician and the theoretical physicist. The former is satisfied with a formula or even an existence theorem. The latter does not consider either an answer at all, until the work of numerical computation has been taken to a stage where comparison with observation is possible. No theoretical physicist is the worse for knowing some pure mathematics that he has never had occasion to use; but he is the worse if the characteristic outlook of the pure mathematician leads him to overemphasize the importance of the mathematics at the expense of an understanding of why the problems are of interest.

Eureka, **4.**

The Euclidean Spook

By D. B. Scott

Practically no important English institutions have been designed for the purposes they serve. They were intended for the quite different needs of an earlier age, and have assumed their present form, which may well be rather different from the original one, after a long process of minor changes. Our educational system is not exceptional in this respect; this is particularly noticeable in the secondary schools. Their course of instruction is not guided by any clear conception of what our society requires for the education of its citizens, but has developed gradually in a rather unsystematic manner. The geometry that is taught there is a good illustration of this, and the purpose of this article is to examine briefly how the present state of geometrical teaching came about, to what extent it justifies the claims made on its behalf, and how far it fits in with the needs of a twentieth-century democracy.

We must remember that the inspiration of our secondary schools lay in the classical studies of our old Public Schools and Grammar Schools, and it was therefore natural for the *Elements* of Euclid to form the foundation of geometrical instruction. Yet although later developments have tended to oust the classics from their dominating position, their effect on mathematical teaching has not been so marked. It is true that the modern geometrical text-book is different from those current at the beginning of the century. Some proofs have been altered to conform more with modern ideas of accuracy. The "proofs" of some theorems whose basis was merely traditional (e.g. that the sum of two adjacent angles is two right angles) are not always given, and the order of presentation has frequently been changed. But for all these superficial alterations the purpose of a school geometry course is unchanged. It is simply to give a systematic development of the proofs of the traditional theorems, and practically nothing is taught except with that end in view.

No one can deny the historical and mathematical interest and importance of the Greek geometry. But that is no guarantee of its suitability, even in its present amended form, for the average secondary pupil. We must remember that Greek mathematics was very different from our own. The modern child can do easily arithmetical problems which would have taxed the powers of the best Greek mathematicians. Algebra was not known in the world of the classics, and this tends to be reflected in their geometry (*cf.* the set of theorems on areas of rectangles which really have no place at a late stage of the geometrical course, but which should be thrown in as illustrations at a very early stage in the algebra). To the Greeks, geometry was the only subject in which reason and logic were supreme. To-day this is not the case. For instance, the logic of algebra is readily understood by children who find the logic of geometry incomprehensible, whilst the spread of the teaching of science should make it possible (even if it has not done so yet) for reasoning powers to be developed in connection with other subjects. But in spite of this, geometry is still justified as the way of teaching people to think. It is true that people with some mathematical abilities can be taught this way, though that does not mean it is the only method. But the majority of children do not learn to appreciate the notions of theoretical geometry, just because they have never been given any understanding of what reasoning means. How can they be expected to reason logically about dry matters like points, lines and angles when they have never learnt to reason even about things in which they are interested? It is like expecting children to be able to write without teaching them to read. It is really almost incredible that it should be assumed that although children have to be taught reading, writing and practically everything else, they nevertheless learn to think, which is much harder, simply by the light of nature and the examples put before them in a subject whose aims and methods most of them never properly understand.

Of course I do not propose that an attempt should be made to remedy this by teaching children formal logic or philosophy, but a great deal could be done in an informal way. Much of what is required would fit naturally into the subject known as English grammar, and ought to be taught long before any attempt is made to prove geometrical theorems. (Incidentally, one of the first things that the young school-teacher finds out is that a large proportion of his mathematical periods is spent in teaching English.) Naturally this will require new ideas of what to include in grammar lessons, but it should be possible to liven up the subject. The introduction of puzzles of a logical character, and of simple detective stories in which the class should be asked to work out, not to guess, a solution, are two possible innovations, which would not only provide a training of real value, but should commend themselves readily both to teachers and pupils. There is also a rather neglected branch of school mathematics, which can be useful in this respect, and that is co-ordinate geometry, which is usually called "Graphs" and regarded as algebra. You can teach children a lot about using their heads by discussing the deductions which can be legitimately made from a graph. This is an interesting subject, and by a suitable choice of the material presented, it should be possible to impart a great deal of useful knowledge in a palatable way, while the ability to appreciate the meaning of a graph, or any other form of statistical conjuring trick, is an essential part of the modern citizen's education.

The belief in Euclid which most educationists seem to hold is very easy to explain. Those on whom the treatment has been successful naturally endorse it, while those on whom it has failed find themselves in the same position as the courtiers confronted with the Emperor's famous new clothes. Only when the imposture is exposed can we hope to produce a school geometry in accordance with the needs of the day, for the theoretical development of Euclidean geometry takes all the available time in the normal school life. The first point we must emphasise, however, is that many of the results that are now taught are too valuable to omit, and they must be acquired, at least as experimental truths. But we must also provide the further geometrical training that contemporary society demands, and as soon as we reject the traditional justification of Euclid, it becomes reasonable to give this training priority over the theoretical geometry now taught. The war has exposed the limitations of traditional mathematical teaching, as is evidenced by the introduction of the A.T.C. in the secondary schools. I have not the time or the space to go into details about what is required. But there are one or two obvious suggestions. People need spatial perceptions, and one of the main purposes of geometrical teaching should be to develop them, and they obviously require a knowledge of solid geometry. Nor should geometry be divorced from other subjects. For instance, geography includes a great deal of geometrical matter and, in particular, map reading is a subject which could well be taught, while woodwork (or needlework) is a subject with an obvious geometrical content. It is not possible to say here how all this should be built up into a comprehensive teaching programme. It would certainly need detailed thought and careful planning. But the planning must not be, as previously, based on the idea of modifying the existing situation, but should form an independent whole based on the needs of the present and the future, rather than on the traditions of the past.

Eureka, **5.**

The Faking of Genetical Results

BY PROFESSOR J. B. S. HALDANE

MY FATHER published a number of papers on blood analysis. In the proofs of one of them the following sentence, or something very like it, occurred: "Unless the blood is very thoroughly faked, it will be found that duplicate determinations rarely agree." Every biochemist will sympathise with this opinion. I may add that the verb "to lake," when applied to blood, means to break up the corpuscles so that it becomes transparent.

In genetical work also, duplicates rarely agree unless they are faked. Thus I may mate two brother black mice, both sons of a black father and a white mother, with two white sisters, and one will beget 10 black and 15 white young; the other 15 black and 10 white. To the ingenuous biologist this appears to be a bad agreement. A mathematician will tell him that where the same ratio of black to white is expected in each family, so large a discrepancy (though how best to compare discrepancies is not obvious) will occur in about 26 per cent. of all cases. If the mathematician is a rigorist he will say the same thing a little more accurately in a great many more words.

A biologist who has no mathematical knowledge, and, what is vastly more serious, no scientific honour, will be tempted to fake his results, and say that he got 12 black and 13 white in one family, and 13 black and 12 white in the other. The temptation is generally more subtle. In one of a number of families where equality is expected he gets 19 black and 6 white mice. It looks much more like a ratio of 3 black to 1 white. How is he to explain it? Wasn't that the cage whose door once seemed to be insecurely fastened? Perhaps the female got out for a while or some other mouse got in. Anyway he had better reject the family. The total gives a better fit to expectation if he does so, by the way. Our poor friend has forgotten the binomial theorem. A study of the expansion of $\left(\frac{1+x}{2}\right)^{25}$ would have shown him that as bad a fit or worse would be obtained with a probability of $122073 \cdot 2^{-23}$, or ·0146. There is nothing at all surprising in getting one family as aberrant as this in a set of 20. But he is now on a slippery slope.

He gets his Ph.D. He wants a fellowship, and time is short. But he has been reading *Nature* and noticed two letters* to that journal of which I was joint author, in which I might appear to have hinted at faking by my genetical colleagues. Thoroughly alarmed, he goes to a venal mathematician. Cambridge is full of mathematicians who have been so corrupted by quantum mechanics that they use series which are clearly divergent, and not even proved to be summable. Interrupting such a one in the midst of an orgy or Bhabha and benzedrine, our villain asks for a treatise on faking. "I am trying to reconcile Milne, Born and Dirac, not to mention some facts which don't seem to agree with any of them, or with Eddington," replies the debauchee, "and I feel discontinuous in every interval; but here goes.

"I suppose you know the hypothesis you want to prove. It wouldn't be a bad thing to grow a few mice or flies or parrots or cucumbers or whatever you're supposed to be working on, to see if your hypothesis is anywhere near the facts. Suppose in a given series of families you expect to get four classes of hedgehogs or whatnot with frequencies p_1, p_2, p_3, p_4, and your total is S, I shouldn't advise you to say you got just Sp_1, Sp_2, Sp_3 and Sp_4, or even the nearest whole number. Here is what you'd better do. Say you got A_1, A_2, A_3 and A_4, and evaluate

$$\chi^2 = \frac{(A_1 - Sp_1)^2}{Sp_1} + \frac{(A_2 - Sp_2)^2}{Sp_2} + \ldots$$

Your χ^2 has three degrees of freedom. That is to say you can say you got A_1 red, A_2 green and A_3 blue hedgehogs. But you will then have to say you got $S - A_1 - A_2 - A_3$ purple ones. Hence the expected value of χ^2 is 3, and its standard error is $\sqrt{6}$; so choose your A's so as to give a χ^2 anywhere between 1 and 6. This is called faking of the first order. It isn't really necessary. You might

* U. Philip and J. B. S. Haldane (1939). *Nature*, **143**, p. 334.
Hans Grüneberg and J. B. S. Haldane (1940). *Nature*, **145**, p. 704.

have $p_1 = \frac{9}{16}, p_2 = \frac{3}{16}, p_4 = \frac{1}{16}$ and $A_1 = 9$, $A_2 = A_3 = 3$, $A_4 = 1$. The probability of getting this is $\frac{16!\,3^{24}}{9!\,(3!)^2\,1!\,16^{16}}$, which is only just under ·04. However, it looks better not to get the exact numbers expected, and if you do it on a population of hundreds or thousands you may be caught out.

"Your second order faking is the same sort of thing. Supposing your total is made up of n families, and you say the rth consisted of $a_{r1}, a_{r2}, a_{r3}, a_{r4}$ members of the four classes, s_r in all, you take

$$\frac{(a_{r1} - s_r p_1)^2}{s_r p_1} + \frac{(a_{r2} - s_r p_2)^2}{s_r p_2} + \ldots$$

and sum for all values of r. Your total ought to be somewhere near $3n$. The standard error is $\sqrt{6n}$, and it's better to be too high than too low. A chap called Moewus in Berlin who counted different types of algae (or so he said), got such a magnificent agreement between observed and theoretical results, that if every member of the human race had repeated his work once a month for 10^{12} years, they might expect as good a fit on one occasion (though not with great confidence). So Moewus certainly hadn't done any second order faking. Of course I don't suggest that he did any faking at all. He may have run into one of those theoretically possible miracles, like the monkey typing out the text of Hamlet by mere luck. But I shouldn't have a miracle like that in your fellowship dissertation.

"There is also third order faking. The $4n$ different components of χ^2 should be distributed round their mean in the proper way. That is to say, not merely their mean, but their mean square, cube and so on, should be near the expected values (but not too near). But I shouldn't worry too much about the higher orders. The only examiner who is likely to spot that you haven't done them is Haldane, and he'll probably be interned as a Red before you send your thesis in. Of course you might get R. A. Fisher, which would be quite as bad. So if you are worried about it you'd better come back and see me later."

Man is an orderly animal. He finds it very hard to imitate the disorder of nature. In fact the situation is the exact opposite of what the reader of Paley's *Evidences* might expect. But the problem is an interesting one, because it raises in a sharp and concrete way the question of what is meant by randomness, a question which, I believe, has not been fully worked out. The number of independent numerical criteria of randomness which can be applied increases with the number of observations, but much more slowly, perhaps as its logarithm. The criteria now in use have been developed to search for excessive irregularity, that is to say, unduly bad fit between observation and hypothesis. It does not follow that they are so well adapted to a search for an unduly good fit. Here, I believe, is a real problem for students of probability. Its solution might lead to a better set of axioms for that very far from rigorous but none the less fascinating branch of mathematics.

Eureka, **6**.

Greek Metamathematics

BY N. A. ROUTLEDGE

Achilles and the Tortoise—a Consideration

How often does the deep simplicity and insight of country folk confound the sophistries of the over-educated! It was round about 450 B.C. that a self-taught philosopher pointed out four paradoxes to the frequenters of the academies in the little Greek colony of Elea. Since then everyone with the slightest interest in philosophy or mathematics has been troubled by them and has developed highly ingenious theories for resolving them. But ingenuity has not been accompanied by conclusiveness: indeed, the paradoxes "have probably occasioned more inconclusive disputation than any equal amount of disguised mathematics in history."

Three of Zeno's posers have been dealt with quite reasonably, but the other, concerning Achilles and the Tortoise, has never received a treatment that did not leave at the back of the mind a nasty feeling that the solver has been a little *too* clever. The argument is:

Achilles runs ten times as fast as the Tortoise.
He gives it a start of 100 yards.
When he has run this the Tortoise is 10 yards ahead.
When he has run this 10 yards the Tortoise is 1 yard ahead.
When he has run this 1 yard the Tortoise is 1/10 yard ahead.
Etc.

(*a*) Thus, according to this argument, Achilles never overtakes the Tortoise.

(*b*) Whereas, of course, we know that he does.

The usual way of treating this is to say that

$$100 + 10 + 1 + 1/10 + 1/100 + \ldots = 111\tfrac{1}{9}$$

and that Achilles overtakes the Tortoise after going $111\frac{1}{9}$ yards, but one cannot sum an infinite series in a finite length of time. This sounds, on a first hearing, as if it disposes of the contradiction, but the more one looks at it the less it seems to do so, and if one tries to rewrite Zeno's proof, one sees that the remarks have no bearing on the problem at all.

The complete solution introduces metamathematics—arguments about arguments. Metamathematics has been developed almost entirely in the last 50 years or so, and has yielded many startling and important results. The most comforting is the solution of Hilbert's *Entscheidungsproblem*: it has been demonstrated that no machine can deal with all mathematics. The most tiresome results concern unsolvability: one takes a logical system and shows that a certain statement in it cannot be shown to be true and cannot be shown to be false. Zeno's paradox in its correct form is precisely of this kind.

To resolve the paradox we merely alter the statement (*a*) to:

Thus, no argument of this kind, however long we continue it, will ever lead us to the conclusion that Achilles overtakes the Tortoise.

Does this now conflict with the statement (*b*)? Only if one argues:

Since our argument cannot show that Achilles overtakes the Tortoise, it must be true that he does not.

We can only assert this if we are sure that every statement can be demonstrated true or else shown false by such arguments. But this is not so, for these arguments allow us to say nothing concerning, for example, where the Tortoise is when Achilles has gone 112 yards.

The paradox has thus vanished.

We can introduce a formal logical system:

The symbols used will be R, (,) , ; , 0. I shall use the abbreviations:

1 for 00
2 for 000
3 for 0000
. . .
n for $n + 1$ zeros in a row.

I have just one axiom: $\quad R(0 ; 1)$

and just one rule of proof:

From $R(n ; m)$ we may conclude $R(n + 1 ; m + 1)$.

Then one may easily see that R(n ; 0) is not provable for any n. If we interpret R(n ; m) as meaning that:

> At the nth stage of Zeno's argument Achilles is $1000/10^m$ yards behind the Tortoise except when $m = 0$, when Achilles is level with the Tortoise,

we see that this system formalises Zeno's method of argument.

The really thrilling thing is to see how near to discovering metamathematics the Greeks were, and it is amusing to speculate what the trend of history would have been had they done so.

Eureka, **13**.

Professor A. S. Besicovitch

By M. F. A.

If, at some time during the last war, you had been wandering through the Great Court of Trinity on a summer afternoon, you might have been privileged to see a distinguished-looking man mowing the lawns in his shirt sleeves. That would have been Abram Samoilovitch Besicovitch, F.R.S.

To the world at large Besicovitch passes for a Russian, but in actual fact he comes from a small and little-known people called the Karaites, a Jewish sect of Turkish origin who are to be found in scattered communities as far apart as Egypt in the south and Finland in the north. Most of them, however, inhabit the area round the Black Sea, and it was there that Besicovitch spent the early years of his life. He himself tells the story of how as a young boy he used to go down to the docks at a small town on the Sea of Azov and ask the English sailors to help him read his English books. On one such occasion he attracted the attention of the captain himself and spent the rest of the time on board as the captain's guest, with a corresponding improvement in the standard of English.

Along with one of his brothers he read mathematics at the University of St. Petersburg, and it was there that he started his academic career, working under the famous mathematician A. A. Markoff on Probability Theory and lecturing, some may be surprised to hear, in Applied Mathematics. With the advent of the revolution life became increasingly difficult in St. Petersburg, and in 1917 he moved to the new University of Perm, in the Urals, where conditions were better but still not really conducive to mathematical research. The rigours of the climate necessitated sitting in a large sack as the only means of keeping warm, and the isolation from the mathematical world was an even more serious obstacle as most of the books in the University were of 1850 vintage and periodicals were an extreme rarity. Despite all these difficulties Besicovitch, during this period, did some work on Real Functions and solved the famous problem of Kakeye. In non-technical language his solution might be described by saying that in order to reverse your car (assumed infinitely thin) you require no room at all, though unfortunately you will have to go off to infinity in an infinite number of directions.

In 1920 he returned to his old University and stayed there until, five years later, he finally left Russia on a Rockefeller research grant and went to Copenhagen. Oxford then claimed him for a year, but fortunately he found his way to Cambridge shortly after, and has remained here ever since.

An analyst throughout his life, he has worked on such topics as Sets of Points, Measure, Real Functions and Parametric Surfaces, not forgetting occasional incursions into other fields such as Number Theory. Perhaps typical of the sort of work he has done is the pathological surface he produced which showed that the definition of area current at the time was completely inadequate. It was the diagram of this surface with its striking similarity to a system of pipes which gave rise to the rumour that he started life as a plumber. Another of his results (though here he was anticipated by Brouwer) was to show that there was no proper equivalent of the four-colour problem in three dimensions, for, even with the obvious restriction to convex polyhedra, the number of colours required turned out to be infinite. With these examples in mind it is not difficult to see why the numbers 0 and ∞ were once described as typical Besicovitch numbers.

Original in other fields as well as mathematics, his leisure hours are spent, not in the orthodox academic pursuit of mountain peaks, but in the quiter joys of long-distance swimming, and the Channel rather than Everest is his goal. Recently he caused a minor sensation at the Canadian Mathematical Seminar by swimming a mile between lectures. It was there also that he produced his famous card game which has rules so simple that it is played by Russian peasants, but yet resembles Chess in the skill and subtlety of its play. He offered two dollars to anyone who could defeat him, and though some of the younger mathematicians tried hard for a whole month, he returned unbeaten, to the great satisfaction, no doubt, of the Chancellor of the Exchequer.

Since his election to the Rouse Ball chair in 1950 he has, of course, given up supervising undergraduates, and those who were fortunate enough to work for him in earlier days will realise what a loss this is to succeeding years of Trinity freshmen. His supervisions were definitely an experience in themselves; one climbed his stairs in a state of trepidation, contemplating sadly the fate that awaited one's efforts on last week's paper, and wondering what deceptive little problems he would produce this week, problems that charmed by their simplicity yet obstinately refused to be solved. But the rigours of the mathematical instruction were always mitigated by his essential kindness. On one occasion, in order to cheer the despondent pupil, he went so far as to confess that he had never really understood the whole question of pole and polar in elementary geometry.

It certainly cannot be said of many professors that they take such a friendly interest in the undergraduate world as Besicovitch. One of his favourite past-times, he says, is going for walks with undergraduates; indeed, so keen is he on these walks that he once assured his young companions that the portending blizzard was, in Russian eyes, a sign of mild weather and need not deter them from their walk. On the occasion of the Commemoration dinner with the subsequent intermingling of High and Low Tables, he is always to be seen in the centre of an animated group, relating some of his popular anecdotes and signing menus for the young autograph hunters. In fact, despite all the minor eccentricities and characteristics which are almost *de rigueur* in a professor of mathematics, the more lasting impressions one carries away are of his extreme amiability and his keen sense of humour.

Eureka, **15.**

A certain distinguished applied mathematician was heard to assert in his lectures: "And so the problem can now be solved without any mathematics at all—just the use of group theory!"

Eureka, **15.**

And thei that bee dulle witted, and yet be instructed and exercised in it (Arithmetike), though thei gette nothyng els, yet this shall thei all obtain, that thei shall bee moare sharpe witted than thei were before.

ROBERT RECORDE, *The Whetstone of Witte*, and *Eureka*, **7.**

Geometer: One who studies geometry; a caterpillar.

Shorter Oxford Dictionary, and *Eureka*, **15.**

Why are Series Musical?

ASKS BLANCHE DESCARTES

MOST mathematicians know the theory of the game of Nim, described in books on mathematical recreations. But few seems to be aware of Dr. P. M. Grundy's remarkable generalisation, published in *Eureka*, **2**, 6–8 (1939). Consider a game Γ in which 2 players move alternately, and the last player wins (moving to a "terminal position"). Define inductively a function G(P) of the position P [Ω(P) in Grundy's Paper] as follows:—

(*a*) if P is terminal, $G(P) = 0$,

(*b*) if there are permitted moves from P to Q, from P to R, from P to S, and so on, then G(P) is the least non-negative integer different from all of G(Q), G(R), G(S), etc.

It follows that if $0 \leqslant r < G(P)$ there is a move from P to some R with $G(R) = r$, but no move to any position U with $G(U) = G(P)$. If positions P with $G(P) = 0$ are called "safe," the winning strategy is to move always to a safe position: either this is terminal, and wins immediately, or the opponent moves to an unsafe position and the cycle repeats.

Now imagine the players engaging in a "simultaneous display" of k games $\Gamma_1, \Gamma_2, \ldots \Gamma_k$ of this sort, the rule being that each player in turn makes a move in one and only one game, or if he cannot move in any game he loses. Let $P_1, P_2, \ldots P_k$ be the positions in the respective games $\Gamma_1, \Gamma_2, \ldots \Gamma_k$. Then Grundy's Theorem states that—

(i) this combined position is safe if and only if k heaps of $G(P_1)$, $G(P_2), \ldots G(P_k)$ counters respectively form a safe combination in Nim,

(ii) more generally, the G function of the combined position is the "nim-sum" of the separate $G(P_s)$ (i.e. obtained by writing the $G(P_s)$ in the scale of 2 and adding columns mod 2).

For no player can gain any advantage by moving so as to increase any $G(P_s)$, as the opponent can restore the *status quo*. And if only decreases in $G(P_s)$ are considered, the game is identical with Nim, thus proving assertion (i). Therefore $G(P) = g$ if and only if the combined position (P, P′) is safe, where $G(P') = g$. From that (ii) follows fairly readily.

It follows that we can analyse any such combined game completely, provided that we can find the $G(P_s)$ for the component positions. Nim is an example; a heap H_x of x counters constitutes a component position, since each player in turn alters one heap only, and $G(H_x) = x$. Many variants of Nim are similarly analysed. Less trivial is Grundy's game, in which any one heap is divided into two unequal (non-empty) parts. Thus heaps of 1, 2, are terminal, with $G = 0$, a heap of 3 can only be divided into $2 + 1$, which is terminal, so $G(H_3) = 1$. Generally $G(H_x)$ in Grundy's game is the least integer $\geqslant 0$ different from all nim-sums of $G(H_y)$ and $G(H_{x-y})$, $0 < y < \frac{1}{2}x$. The series goes

$x =$	0	1	2	3	4	5	6	7	8	9	10	11	12
$G(H_x) =$	0	0	0	1	0	2	1	0	2	1	0	2	1

continuing with 3, 2, 1, 3, 2, 4, 3, 0, 4, 3, 0, 4, 3, 0, 4, 1, 2, 3, 1, 2, 4, 1, 2, 4, 1, 2, . . . This curious "somewhat periodic series" seems to be trying to have period 3, but with jumps continually occurring. Mr. Richard K. Guy confirmed this up to $x = 300$. He suggested that it might be played on a piano, taking 0 to be middle C, 1 = D, 2 = E, etc. The inner meaning then became evident:

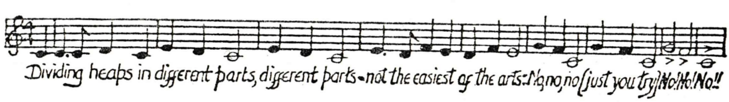


FIG. 1.

Guy also worked with rows R_x of x counters, in which certain sets of consecutive counters could be extracted (thus possibly leaving two shorter rows, one each side of the extracted set). In his "**·6**" game, any one counter can be removed, except an R_1 (= a single counter standing on its own). The $G(R_x)$ series ($x = 1, 2, \ldots$) is a waltz (N.B. some notes span two bars):

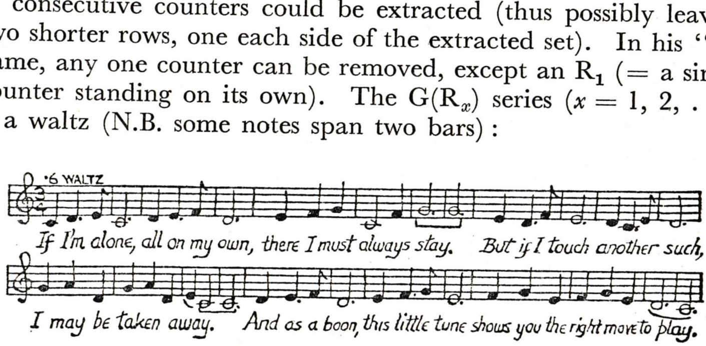


FIG. 2.

But at this point the tune completely broke down. I asked Guy if he could think of any reason for that: he said, "Yes, an error

I made in the calculation." After correction the waltz proceeds:

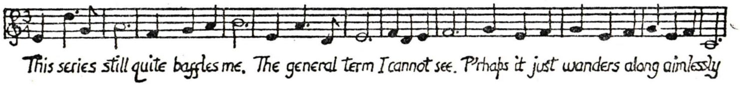


FIG. 3.

This tries to be periodic with period 26, but jumps keep appearing. Many other such games give tuneful, somewhat periodic series, for no evident reason. Guy discovered two curious exceptions: his "**·4**," remove 1 counter not at the end of a row, has exact period 34 for $x \geqslant 54$, and Kayles, remove 1 or 2 adjacent counters, has exact period 12 for $x \geqslant 71$. Thus these games have a complete analysis. But generally it might be helpful to bring in a professional musician to study number theory. Perhaps a thorough study of Fermat's Last Theorem will uncover the Lost Chord. After all, why not?

Eureka, **16.**

A Pathological Function

BY J. D. ROBERTS

WE DEFINE a function $F(x)$ which takes every real value in every interval.

Express the fractional part of x as a binary "decimal" (which may or may not terminate) $0.x_1x_2x_3 \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$

We denote the sequence $0111 \ldots$ (n 1's)0 by S_n and define

$f_n = 0$ if S_n occurs a finite number of times in (1)

$f_n = 1$ if S_n occurs an infinite number of times

and $f(x) = 0.f_1f_2f_3 \ldots$ (another binary decimal).

Now take any interval of x (as small as we like). We can find $N.x_1x_2 \ldots x_r$ such that with these fixed x is bound to lie in the interval. We can now make $f(x)$ take any value $0.f_1f_2 \ldots$ between 0 and 1. For consider the sequence $n_1, n_2, n_3, \ldots$ (which may terminate) of all the n's for which $f_n = 1$, and write

$$x = N.x_1x_2 \ldots x_r;\ S_{n1};\ S_{n1}, S_{n2};\ S_{n1}, S_{n2}, S_{n3};\ \ldots$$

which contains all S_{ni} an infinite number of times and all other S_n a finite number of times.

By considering $F(x) = \tan \pi \{f(x) - \frac{1}{2}\}$ we have a function which takes every real value in every interval of x.

Eureka, **20.**

Our Founder

By H. T. Croft

Archimedes of Syracuse was the son of Pheidias the astronomer, and on intimate terms with, if not related to, King Hieron and his son Gelon. He spent some of his life in Alexandria, and was friendly with Conon of Samos and Eratosthenes; then returned to Syracuse for a life devoted to mathematical research. He perished in 212 B.C. (at age 75, according to Tzetzes) in the sack of Syracuse.

Stories of other details of his life, culled from many sources, are somewhat dubious. No authenticated picture remains, in spite of three (totally different) purported portraitures in classical works of the last century. The only contemporary biography is not extant.

Tales of his preoccupied abstraction—drawing diagrams in ashes, or in oil when anointing himself, and forgetfulness of food—remind us irresistibly of Newton's going out in a fit of absentmindedness without his trousers. He died as he had lived, deep in mathematical contemplation. Several authors give variously garbled accounts, the most picturesque being that, though Marcellus the Roman commander wished him to be spared, a common soldier, enraged by the great man's request to "Stand away, fellow, from my diagram," dispatched him. As he had asked, his discovery of the surfaces of the sphere and cylinder was depicted on his tombstone, which was later found in a dilapidated state and restored by Cicero when quaestor in Sicily.

His mechanical achievements include the water-screw, invented in Egypt for irrigational purposes and used for pumping from mines or ship-holds, and a very accurate model of the planetary system demonstrating eclipses. Some of his inventions were very effective during the siege of Syracuse—catapults of variable range and other machines discharging showers of missiles, and crane-like grappling contrivances which seized the prows of ships and thus played "pitch-and-toss" with them. The Romans were in such abject terror that "if they did but see a piece of rope or wood projecting above the wall, they would cry 'there it is again,' declaring that Archimedes was setting some engine in motion against them, and would turn their backs and run away, insomuch that Marcellus desisted from all conflicts and assaults, putting all his hope in a long siege" (Plutarch). The story that he fired the Roman ships by use of concave burning-glasses and mirrors is very doubtful, being first recorded in Lucian, 300 years later.

When Hieron asked for a practical demonstration of a great weight moved by a small force, in connection with his famous utterance "δός μοι ποῦ στῶ καὶ κινῶ τὴν γῆν" (Give me a place to stand on, and I can move the Earth), he drew a loaded ship safely and smoothly along with a compound pulley or, according to another account, a helix, a machine with a cogwheel with oblique teeth. Hieron thereupon declared that "from that day forth Archimedes was to be believed in everything that he might say."

Born just before the death of Euclid and 30 years senior to Apollonius of Perga, the "Great Geometer" and last of the three great mathematicians of antiquity, he wrote works with a larger proportion of originality. "It is not possible to find in geometry more difficult and troublesome questions or more simple and lucid explanations." Like most of the ancients, he left little clue as to his method of discovery. He seems "as it were of set purpose to have covered up the traces of his investigation as if he had grudged posterity the secret of his method of inquiry while he wished to extort from them assent to his results" (Wallis). But a manuscript of the "Methods of mechanical theorems," discovered in 1906 in Constantinople and addressed to Eratosthenes, lifts the veil a little.

Other works entitled "On the equilibrium of planes," "On the quadrature of the parabola," "On conoids and spheroids," "On floating bodies," "On the measurement of a circle," "The Sand-reckoner" and a collection of lemmas indirectly due to him are still extant. Lost works are thought to refer to polyhedra, balances and levers, centres of gravity, the calendar, optics, water-clocks and a work entitled "On sphere-making." Arabian writers attribute other works to him.

His main achievements were: quadrature of the parabola, finding surfaces and volumes of spheres, segments of spheres and segments of quadrics of revolution, the approximation $3\frac{1}{7} > \pi > 3\frac{10}{71}$, the invention of a number-scale up to 10 to the power 8.10^{10} (in the Sand-reckoner), geometrical solution of some cubic equations, a method of finding square roots of non-squares, and the whole science of hydrostatics even up to determining the positions of equilibrium and stability of floating segments of a paraboloid. He was also much occupied by astronomy—Livy calls him "*unicus spectator caeli siderumque.*" He is further credited with authorship of the "cattle-problem," which involves eight unknowns and the solution of which has 12 or 206545 digits according to how an ambiguous statement is interpreted. The "*loculus Archimedius,*" a puzzle of 14 shapes fitting together to form a square, is now thought due to him, although the phrase "πρόβλημα Ἀρχιμήδειον" was simply a proverbial expression for something very difficult.

He regarded his ingenious mechanical investions simply as "diversions of geometry at play" and "he possessed so high a spirit, so profound a soul, and such treasures of scientific knowledge that, though these inventions had obtained for him the renown of more than human sagacity, he yet would not deign to leave behind him any written work on such subjects, but, regarding as ignoble and sordid the business of mechanics and every sort of art which is directed to use and profit, he placed his whole ambition in those speculations in whose beauty and subtlety there is no admixture of the common needs of life" (Plutarch).

Eureka, **21**.

Vipers, Logs and All That

By G. J. S. Ross

It is widely believed that the only mathematician in the Bible was Noah. Nobody else would have had a hope of passing the Eleven Plus. Admittedly, Moses' Book of Numbers is frankly disappointing, but I hope to show in this article that the Bible contains evidence of a higher standard of mathematics than is generally supposed.

Arithmetic if, of course, mentioned most frequently, and we are told that men sometimes worshipped figures.[1] At a very early stage "men began to multiply,"[2] and Abraham was familiar with division.[3] Some writers have pointed out that the arithmetic in Ezra[4] is faulty, but this is explained where it reads "certain additions were made of thin work."[5] The approximation for π is reasonable,[6] considering the fact that Moses destroyed the tables,[7] which were not replaced until Solomon's time.[8] Elsewhere we read "he shall not extract the root thereof,"[9] and "we wrestle against powers."[10]

The first attempts at Geometry were, of course, Euclidean. We read that "great rulers were brought down,"[11] "from Syracuse they fetched a compass,"[12] and Noah constructed an arc[13] and Ezekiel described a line.[14] Further progress was made when they took axes,[15] culminating in David's success with the calculus.[16] David, incidentally, was the first to refuse to accept what he had not proved.[17] St. Paul was familiar with four dimensions,[18] and Joshua continued with the arc long a Jordan path.[19]

Algebra, although thought to be an invention of the Arabs, was only too familiar to the Jews. For instance, Moses gives instructions about a matrix[20] and Ezekiel knew enough about rings to describe them as "dreadful."[21] Peter was kept half the night by four quaternions,[22] and the Jews were described as "a generation seeking after a sign."[23]

"As for the Pure, his work is right" said the writer of Proverbs,[24] and this attitude is reflected in the few existing references to Applied Mathematics. "I have seen thy abominations in the Fields" cried Jeremiah,[25] and the Psalmist complained "Thou hast afflicted me with all thy Waves."[26] Later the Father of Publius was "sick of the bloody Flux."[27]

It is easy to understand why they disliked mathematics. Apart from the deacons "who purchase to themselves a good Degree,"[28] they had to be examined, as was St. Paul.[29] We know that Elisha passed,[30] and Solomon was able to answer all the questions,[31] but Peter was much troubled when he saw the sheet,[32] and Job cried "My kinsfolk have failed, and my friends."[33] Perhaps Johoiakim was an examiner, for "when he had read three or four pages he cast it into the fire."[34] As for St. John, all that he knew was "the Second woe is past, the Third cometh."[35]

Eureka, **22.**

[1] Acts vii. 43. [2] Gen. vi. 1. [3] Gen. xv. 10. [4] Ezra ii. [5] 1 Kings vii. 29. [6] 2 Chron. iv. 2. [7] Exod. xxxii. 19. [8] 2 Chron. iv. 8. [9] Ezek. xvii. 9. [10] Eph. vi. 12. [11] Ps. 136. 17. [12] Acts xxviii. 13. [13] Gen. vi. (archaic spelling). [14] Ezek. xl. [15] 1 Sam. xiii. 21. [16] 1 Sam. xvii. [17] 1 Sam. xvii. 39. [18] Eph. iii. 18. [19] Joshua iii. [20] Exod. xxxiv. 19. [21] Ezek. i. 18. [22] Acts xii. 4. [23] Math. xvi. 4. [24] Prov. xxi. 8. [25] Jer. xiii. 27. [26] Ps. 88. 7. [27] Acts xxviii. 8. [28] 1 Tim. iii. 13. [29] Acts xxviii. 18. [30] 2 Kings iv. 8. [31] 2 Chron. ix. 2. [32] Acts xi. [33] Job xix. 14. [34] Jer. xxxvi. 23. [35] Rev. xi. 14.

Fermat's Last Theorem

FIVE people make the following statements:—

(1) *Either* (*a*) 3's statement is false and 4's statement is true.
Or (*b*) 2's and 5's are both false.

(2) *Either* (*a*) 4's statement is false *or* 3's is false.
Or (*b*) 1's and 5's are both false.

(3) 2's statement is true *or* 4's and 5's are both true.
Moreover, *either* 5's is false *or* 4's is true.

(4) 3's statement is false *or* 1's is true.

(5) Fermat's last theorem is true.

Which of these statements are true and which false? It will be found on trial that there is only one possibility. Thus, prove or disprove Fermat's last theorem.

Eureka, **9.**

A Very Short History of Mathematics

By W. W. O. Schlesinger and A. R. Curtis

This paper was read to the Adams Society (St. John's College Mathematical Society) at their 25th anniversary dinner, Michaelmas Term, 1948.

Mathematics is very much older than History, which begins* in + 1066, as is well known; for the first mathematician of any note was a Greek named Zeno, who was born in −494, just 1,559 years earlier. Zeno is memorable for proving three theorems: (i) that motion is impossible; (ii) that Achilles can never catch the tortoise (he failed to notice that this follows from his first theorem); and (iii) that half the time may be equal to double the time. This was not considered a very good start by the other Greeks, so they turned their attention to Geometry.

Euclid, about −300, invented Geometry, including Pythagoras' theorem which is how it got the name. He also invented parallel lines, which have really been of more use to railways than to mathematicians. Most people already know more about Euclid than we do.

Archimedes (−286 to −211) is very memorable for taking a bath. Unfortunately he forgot to get dressed afterwards, in spite of his principles.

From this time onward there was an open interval, the other end point of which was Descartes (1596 to 1650), who was divinely inspired to invent analytical geometry, and was once found sitting inside a stove to keep himself warm. He also discovered that he existed, and, moreover, he was able to prove it.

Newton (1642 to 1727) was very memorable indeed, chiefly for having just missed living in St. John's. To console himself he invented the Calculus.

Newton is also memorable for having been admired by Taylor, who invented Maclaurin's series and admired Newton. However, Taylor lived in St. John's and so was luckier than Newton.

The next important mathematician is the Bernoullis. In spite of his having invented numbers, nobody knows how many of him there were, and he lived all over the century. He was called Nicholas, Jacob and John, and one of him was called Daniel.

Euler (1707 to 1783), Langrange (1736 to 1813), and Laplace (1749 to 1827) are all famous for inventing equations. Only one of Laplace's equations is well known, but this is enough for anyone.

* See Sellars and Yeatman, "1066 and all that," or any similar standard work.

It makes electricity and hydrodynamics much easier for people who don't have to solve it. Euler and Lagrange both went about varying things, which caused the calculus of variations. This was both memorable and regrettable.

Gauss invented so many things that it just isn't true. These included the magnetism of the earth, the theory of equations, Cauchy's theorem and the Cauchy-Riemann equations. In fact, whenever anyone invented anything in the first half of the 19th century, Gauss had invented it twenty years earlier, and was still alive to tell him so. He was born in 1777, died in 1855, and lived all the years in between. He was very memorable, and a good thing.

Cauchy's theorem is very important, but is much harder to prove now than it was when Gauss invented it.

Lobatchewski (1793 to 1856) must have failed an examination in geometry when he was at school, for he made things harder for everyone by inventing non-Euclidean geometry—just to get his revenge, of course. This was especially bad for the railways, since it made parallel lines so much more difficult.

Hamilton (1805 to 1865) was an Irishman . When he had learnt 13 languages before he had left school, he decided that there was no future in this, and look up mathematics. He invented Hamilton's priniciple, the Hamiltonian, the Hamilton-Jacobi theorem, and the Hamilton-Cayley theorem, but not the Hamilton Academicals. Towards the end of his life he also invented quaternions, but nobody except himself ever fell in love with them.

Weierstrass (1815 to 1897) is memorable because of Sonja Kowalewski (1850 to 1891), who, of course, is memorable because of Weierstrass. He said that if you put infinitely many things into a small space, some of them would be pretty close together.

The most memorable of all mathematicians was John Couch Adams (1819 to 1890). He had the good fortune to live in St. John's, and was named after this society. He discovered Neptune just after Leverrier, and would have discovered it before if the Astronomer Royal had kept his eyes open.

Charles Lutwidge Dodgson was a minor Oxford mathematician who must not be confused with Lewis Carroll, whom he impersonated when sending copies of his works to Queen Victoria. They lived contemporaneously.

The chief problem treated by Carroll was that of the Cheshire cat. His treatment is essentially unsound, however, since he says: "... this time the cat vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone."* It is obvious that, by the time the

*Lewis Carroll, *Alice's Adventures in Wonderland.*

tail had disappeared, the cat would be a Manx cat. This is a contradiction, since it was a Cheshire cat, by hypothesis. Carroll also discussed the increased angular velocity of the world if everybody minded his own business.

Riemann (1826 to 1866) invented the tensor calculus, and thus caused the theory of relativity.

In 1895 Bertrand Russell stated the following theorem: the class of all classes which are not members of themselves is either a member of itself or not. Whichever it is, it is the other. This a contradiction, and the end of mathematics.

Eureka, **12.**

A Song . . .
Against Mathematicians

Of all the lunatic professions which are practised on this earth
Mathematics is the craziest, and has been from its birth.
Take a look at its practitioners, examine each in turn,
And watch them going further round the bend as more they learn:
The proud arithmetician, who can contemplate infinities
With crude familiarity, and not see what a sin it is,
(Infinities of such a size, he calls a set equal if
The greatest of their differences is small compared to aleph;)
The negligent dynamicist, Procrustes of equations,
Ignoring any higher power which foils his machinations;
If there is any man with more impiety than him, it
Is the unrepentant analyst, proceeding to the limit.
There is a young geometer, aged 23 mod 40;
His views on art are trivial, his views on life are naughty;
The only scheme of government this student can envisage is
To couple all constituents in independent syzygies.
Ye narrow minded bachelors, whose one joy is to figure!
Were you cut up and randomised, set down in utmost rigour,
Your singularities enclosed in everlasting cedar,
Who would lament your absence? . . .

But I leave that to the reader.

D. HANDSCOMB.
Eureka, **17.**

Problems Drive Solutions

Five minutes were allowed for each question. The average marks (out of ten) obtained by the 24 pairs competing are given in brackets. The highest total was 48.

A. (5·6) One point given per prime.

B. (0·1) 3 p.m. ($3 \times 51 + 27$ minutes past noon).

C. (5·2) (*a*) 18, 20 (prime numbers plus one).
(*b*) 18 (81 reversed).
(*c*) 53104 ($= 231^2 - 16^2$).
(*d*) 74 (sum of the fifth terms of the other three series). [Alternative solution: 163 (second differences are 21).]

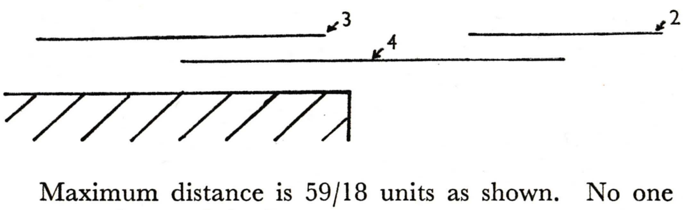
D. (1·3)

6 pt.	10 pt.	15 pt.
0	0	15
0	10	5
0	0	5
0	5	0
6	5	0
0	5	6
6	5	6
1	10	6
1	1	15

No one discovered this! However, four pairs gained half marks for a solution one move longer.

E. (0·5) 19. Again, no fully correct solutions.

F. (0·0)



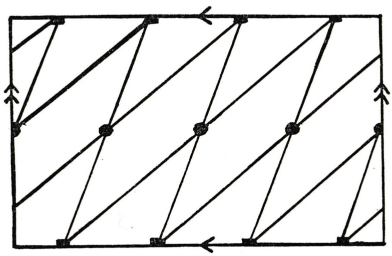
Maximum distance is 59/18 units as shown. No one discovered any of the best FOUR arrangements.

G. (0·4) a = b = c = d = 11.

H. (4·5) (*a*) 1/3, (*b*) 1/3, (*c*) 2/3, (*d*) 1/2.

I. (5·5) 6.

J. (3·6) Four cities, as in the diagram below:—



K. (·) 13.

L. (0·1) 8 years. Sister's present age, 5, satisfies $8/3 < 5 < 8\frac{8}{9}$.

Book Reviews

Numerical Methods, Vol. I. By B. NOBLE. (Oliver & Boyd Ltd.) 10*s*. 6*d*.

This is the first of two volumes designed to give an elementary exposition of certain basic ideas in numerical analysis in a manner suitable for applied scientists. The author has chosen to consider in some detail a limited selection of typical numerical methods with the object of illustrating the general underlying principles. Particularly pleasing is the emphasis laid throughout upon the estimation and practical control of errors.

The first chapter is concerned with the various types of error and the manner in which they occur in calculations, with particular regard to rounding-off, and includes a discussion of the possible ways in which the accumulation of rounding errors may be minimized when evaluating formulae on desk machines. This is followed by a general analysis of iterative procedures for solving algebraic and transcendental equations. The solution of systems of linear algebraic equations together with the evaluation of eigenvalues and eigenvectors occupy the second half of the book, and here the author has succeeded in covering the important topics remarkably well. One particularly interesting chapter is that concerned with elementary programming, in which a simple language, not based on any specific machine but typical of many autocodes currently in use, is developed and subsequently used to indicate how various computational procedures may be implemented on a digital computer. The text is supplemented by a valuable selection of exercises in each chapter.

As the author remarks, the book is both supplemented and complemented by the already well known *Modern Computing Methods* which is somewhat more of a working manual, so that in conjunction these books will provide an excellent grounding for any scientist or engineer engaged in non-trivial computations. The reviewer looks forward with pleasant anticipation to volume 2, which will deal with the numerical solution of ordinary and partial differential equations.

J. OLIVER.

Fallacies in Mathematics. By E. A. MAXWELL. (Cambridge University Press.) 6*s*. 6*d*.

Some twenty-eight "proofs" of various "theorems" and half that number of howlers (false derivations of correct conclusions to given premises) from several branches of mathematics are collected together in this entertaining and instructive work. Dr. Maxwell's lucidly written account of the proofs and their errors is a delight to read, and the Syndics of the Cambridge University Press are to be warmly commended for issuing the work as a paperback (originally published in 1959). D. J. DALEY.

Russian-English Mathematical Vocabulary. By J. BURLAK and K. BROOKE. (Oliver & Boyd Ltd.) 21*s*.

The short guide to reading Russian which precedes the vocabulary contains a very concise summary of Russian grammar and word formation. This claims to be only "a key to the Russian 'code' rather than a Russian grammar," and the person who wishes to read Russian fluently will need to use a more comprehensive grammar, though this might be an adequate introduction. There is a strong emphasis on word formation, and the tables of prefixes, suffixes and final letters are useful for reference.

The vocabulary sets out to provide a complete coverage of terms used in pure mathematics and of most general terms which occur in mathematical works. It has about 11,000 entries with a liberal supply of common phrases and special usages. There are ample cross-references for irregular parts of verbs, and stress is indicated. This is an excellent dictionary for a person with some knowledge of Russian who wishes to read mathematics.

J. M. O. SPEAKMAN.

New Directions in Mathematics. By JOHN G. KEMENY, ROBIN ROBINSON and ROBERT W. RITCHIE. (Prentice-Hall.)

A conference entitled "New Directions in Mathematics" was held at Dartmouth, U.S.A., in 1961, and this 120-page book is an edited transcript of the proceedings. There are four sections: Secondary School, College, Applied and Pure Mathematics; each section consists of three or four prepared speeches, followed by spontaneous discussion.

As is common in such a situation, the spoken word, when written down, is not nearly so effective; what may have been serious discussion all too often comes through as an attempt to score a debating point. And was it really necessary to include everything—"Professor, do you have any further announcements? If not, this meeting is adjourned"?

Collectively, the many well-known mathematicians (Buck, Kac, Kaplansky are amongst them) produced several interesting prophesies and ideas. However, no attempt was made to integrate the suggestions, and at various points the speakers appeared to be at cross-purposes. Informative and relevant this book may well be, but it is hardly worth hard covers. J. HAIGH.

Elements of Mathematical Logic. By P. S. Novikov. Translated by L. F. Boron. With a Preface and Notes by R. L. Goodstein. (Oliver & Boyd Ltd.) 50*s*.

This book is a translation of a very readable Russian book first published in 1959. In his foreword Novikov very aptly describes it as a textbook "to give a discussion of the fundamentals of mathematical logic in the most accessible form possible." It is on the whole very clear (despite a number of sentences which one hopes were more intelligible in the original Russian) and is always interesting. The implications of almost every theorem proved are discussed, and the reader sees why the theorem is necessary and what will follow from it.

There are six chapters and an Introduction. In the Introduction Novikov gives a general description of the problems and methods of mathemetical logic and the reasons for studying them. The first chapter presents the propositional calculus in a non-formal way; the second chapter develops it as a formal system and introduces the reader to metalogical reasoning. The following two chapters repeat this development for the predicate calculus. The fifth chapter on axiomatic arithmetic is basically an extension of the discussion of the predicate calculus to the system of arithmetic. In the final chapter Novikov proves the consistency of a system of restricted arithmetic.

This book is only an introduction to the field of mathematical logic, but as an introduction to the fundamentals of the subject, it is excellent.

P. B. GOLDSTEIN.

Introduction to Mathematical Logic. By Elliott Mendelson. (Van Nostrand Co. Ltd.) 54*s*.

This book is a general introduction to the field of mathemetical logic. It is a concise survey of most of the principal branches of this field. The author assumes no prerequisites other than "a certain amount of experience in abstract mathematical thought."

Philosophical and general discussion are kept to an absolute minimum; the practical aspect of the enquiry dominates the book. Using this approach the author is able to cover a great deal of material, and he does this clearly and concisely. Mendelson begins with a truth-table discussion of the propositional calculus. This is followed by an axiomatic development. The second chapter presents quantification theory. The third chapter develops arithmetic as a formal system, introduces recursive functions and gives a proof of Gödel's incompleteness theorem. In the next chapter Mendelson presents the version of axiomatic set theory as developed by von Neumann, Robinson, Bernays and Gödel. The final chapter is devoted to the problem of effective computability. Mendelson discusses the approaches of Markov, Turing, and Herbrand and Gödel.

There are many exercises throughout the book, some of which are very interesting and point to applications and alternative developments of the subject being considered. These are often valuable in showing the reader new ways in which to view the material at hand.

For the reader with some familiarity with mathematical methods, and especially for one interested in the important results of the field, this book would be a very adequate introduction to mathematical logic.

P. B. GOLDSTEIN.

Introduction to Linear Algebra. By FRANK M. STEWART. (Van Nostrand Co. Ltd.) 58*s*. 6*d*.

This book deserves to become a standard Part I text. It maintains the high tradition of clarity and rigour of Van Nostrand's University Series in Undergraduate Mathematics to which it is the latest addition, and gives a firm motivation for the study of abstract linear spaces.

The reader is aided by three welcome features: an index of symbols used; six appendices, placed at the end of the book to allow the text to be read smoothly, and very illuminating when consulted; and, most important, a summary after each chapter of the most fundamental ideas, definitions and results.

Chapters I and II, although meant as introductory, are quite essential, and must not be skipped; the main abstraction comes in Chapter III. The fourth chapter gives the theory of linear transformations and matrices, with simultaneous linear equations as motivation for both. Then come two independent accounts of the theory of determinants; one from the point of view of multilinear and alternating forms, and the other a more "traditional" treatment. The rest of the book is independent of the first treatment, and the final chapter is about (complex) inner product spaces; the text closes with a proof of the Spectral Theorem.

Both teacher and student will welcome the 200 or so exercises, and the diagrams at judicious points: a student who reads this book will have the firmest of foundations for later work. J. HAIGH.

Number Theory. By J. HUNTER, Ph.D. (Oliver & Boyd Ltd.) 10*s*. 6*d*.

To write a book on Number Theory in less than 150 pages implies of necessity considerable selection of material, and might result in sketchiness of treatment. This, however, Dr. Hunter avoids, and gives a thorough basic introduction to the subject. He emphasises the algebraic basis for the subject, introducing the reader in the first chapter to the concepts of binary operation, equivalence class, group, ring, integral domain, and field, and draws his attention to their occurrence as we proceed with the theory of numbers; for instance, the existence of a primitive root (mod m) is related to the reduced set of residues forming a cyclic group.

In the first five chapters Dr. Hunter defines the natural numbers by Peano's Axioms, and describes their extension to the ring of all integers, and the field of rationals. Factorization and division properties of integers are dealt with in detail, and the theory of congruences leads on to a thorough discussion of primitive roots and quadratic residues. In the final two chapters the reader is introduced to the representation of integers by a binary quadratic form (not necessarily positive definite), with that by the sum of two squares as a special case, and to one or two Diophantine equations such as $x^4 + y^4 = z^2$.

As is to be expected in a book of this size many interesting subjects receive no mention, but those that do are treated with thoroughness, and the examples at the end of each chapter are both interesting and instructive. I. B. T.

Integration, Measure and Probability. By H. R. PITT. (Oliver & Boyd Ltd.) 25*s*.

The author of the first volume of Oliver and Boyd's new series of University Mathematical Monographs sets himself the task of providing "an introduction to the modern theory of probability and the mathematical ideas and techniques on which it is based." The first forty-four pages contain an account of those parts of the theory of sets, measure and integration which the author requires for his subsequent discussion of probability theory. The treatment of the latter includes the standard distributions used in statistics, a brief account of dependence and conditional probability, convergence of sequences of independent random variables, and finally a glimpse of the theory of stochastic processes. The results are given in theorem form throughout, and their proofs are made easier to follow by ample cross-referencing.

From the student's viewpoint it is a pity that there are no exercises set out as such, while a few comments on the "suggestions for further reading" would help the interested reader in continuing this studies. However, the book contains much that is normally given in an introductory course, and it is presented in a readable fashion.

Depending on one's point of view (statistician versus mathematician) one may or may not like the manner in which the author avoids the conventional definition of a random variable by defining the contexts in which the terms "random variable" and "probability" arise. From the first part of the book, as a result, essentially only the theory relating to Lebesgue-Stieltjes integration is used. Such an approach may be satisfactory within the confines of the book under review, but it hardly prepares the student to tackle more extensive treatises like Loève and Doob. D. J. Daley.

The Universal Encyclopedia of Mathematics. (George Allen & Unwin Ltd.) 42*s.*

This work is an English translation, with certain modifications, of a compendium which, in its German version, Meyers Rechurduden, sold over over 200,000 copies in two years. It is aimed at "the Man in the Street, the harassed parent, and the technical or engineering student; or the scientist, engineer or accountant, for whom mathematics have not lost their fascination."

For these purposes the book is well suited: unpretentious definitions are supported by explanations, examples, properties and diagrams. In a work of this nature, whatever has to be omitted would cause some grumbles; this reviewer would have liked more cross-references—for instance, Pascal's and Brianchon's theorems are included, but not linked; and similarly with linear and affine transformations.

In addition to nearly 500 pages of alphabetically arranged "dictionary," we have 100 pages of tables, roots, powers, logarithms, trigonometric, hyperbolic and exponential functions; and a vast quantity of useful formulae, including tables of integrals, expansions of transcendental functions, and vector identities, among many others. Why not buy this book for the family at Christmas? They might let you borrow it sometime. J. Haigh.

Introduction to the Theory of Finite Groups. By Walter Ledermann. (Oliver & Boyd Ltd.) Fifth Edition. 10*s.* 6*d.*

A welcome re-issue, differing only slightly from the fourth edition, of a book which has established itself as a standard Part I text. P. M. Lee.

Continued Fractions. By A. Ya. Khinchin. (University of Chicago Press.) Cloth, 37*s.* 6*d.* Paper, 14*s.*

This is a translation of the third (1961) edition of Khinchin's beautiful introduction to the theory of continued fractions. The main topics covered are the representation of numbers as continued fractions, convergents as best approximations, order of approximation, and the groundwork of the measure theory of continued fractions, including a treatment of various averaging operations connected with their elements. In a book of this length (95 pp.) which assumes very little mathematical knowledge on the part of the reader, only an outline of the subject can be given, but within the limitations he set himself, Khinchin achieved a clarity of exposition which it would be difficult to surpass. P. M. Lee.

Allan Malcolm.

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