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LATE ARRIVAL

The staff of Eureka wish to apologise for the late arrival of this edition, which was caused by printing difficulties beyond their control.

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EURÉKA

THE JOURNAL OF THE ARCHIMEDEANS

NUMBER 37, OCTOBER 1974

EDITOR R.G.E. PINCH

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Editorial

This edition of Eureka is printed in an entirely new manner. Whereas in previous years, the text was sent straight to the printer, who set it into type or composed a lithographic master, this text was transcribed onto a computer file, the computer used to edit and correct the copy, and a copy typed on a flexowriter. This was then used directly as the lithographic master. This has obviously saved much on the printer's costs, with the result that this is the first time for three years that the magazine has made a profit. I hope that in future this system will be seen as the best compromise between cost and quality.

Archimedeans

by Angela Hey (President)

The Michelmas term's meetings were very well attended, especially Dr. Conway's lunch-time talk on "The least uninteresting number". Other lunch meetings included a fallacious argument enthusiastically proposed by Dr. Gough entitled "Swirling whirls without indigestion". Professor Kaldor (since made a life peer) showed little confidence in linear mathematical models applied to economics when he addressed the society at a Friday evening meeting. Dr. B. Thwaites made a return visit to the society in the Lent term. A lucid account of simple groups of small 2-rank was given by Professor Harada (SRC visiting fellow).

The triennial dinner was held in the Michelmas term in Emmanuel College. Lady Jeffreys and Professor Swinnerton-Dyer spoke. A debate between Dr. Conway and the bard, Dr. Reid, proved popular, and it was concluded that one's health did not suffer as a result of mathematics. Besides the annnual visit to Oxford to play games with the "Invariants", a trip to the National Physics Laboratory took place.

Cambridge won the problems drive against the Invariants which was held in Trinity O.C.R..

An afternoon of croquet on Trinity and St. John's backs was the first social event of the Easter term. At the punt party the tradition of throwing the president into the river was revived after a lapse of two years - indeed a precedent has been set for ducking most committee members on this aquatic venture. The ramble went downstream from Jesus lock and then to the Gogs - the party dispersing as usual before converging on Cambridge.

The successful year which has passed could not have been so without the hard work of committee members and I look forward to yet another good year for the Archimedeans.

Mellin Transforms

by K.J. Evans

Mathematics has its folk lore. Theorems are provéd and passed on by word of mouth. The proofs are forgotten but hopefully the point remains. A rich field for folklore is the theory of asymptotic expansions of integrals, for example

$$F(v) = \int dx\ k(v,x) \sim \Sigma\, a\ v$$

One reason for folk lore is that life is too short to read all of mathematics. There are folk lore ideas which the initiated could carefully define. Thus in eq.1 we talk about the dominant region of integration which gives the asymptotic expansion.

One way of finding asymptotic expansions is to transform to a new variable l, in which the large v behaviour of f is represented by the singularity structure of $\hat{f}$ as a function of l. A familiar example is the Laplace transform where poles in the transformed plane correspond to an exponential behaviour in v. There are many other transforms. We shall use the Mellin [ref. 1] because the poles in l correspond to a power behaviour in v.

The Mellin transform is defined by

$$\hat{f}(l) = \int_0^{\infty} dv\ v^{-l-1}\ f(v)$$

The integral in (2) will be defined as a Riemann integral only for a restricted region of l. f(l) is defined for other values of l by analytic continuation from the region in which the integral is defined. There is the inverse formula

$$2\pi i f(v) = \int_{k-i\infty}^{k+i\infty} dl\ v^{l}\hat{f}(l)$$

The problem is to decide what k is! The poles just to the left of the contour in (3) give the large v behaviour of f according to the formulae

$$\hat{f}(l) = (l - a)^{-b-1} \quad \Longleftrightarrow \quad f(v) = v\ \log^{b}(v)/b!$$

Let us apply this technique to F in eq.1. Our procedure will be to Mellin transform the integrand k(v,x) and then see for what values of l the dx integration diverges. If the integral diverges because the integrand becomes singular, the dominant region of integration will be the neighbourhood of this singularity.

Here is an example

$$F(v) = \int_0^1 dx\ g(x)\exp(ixv)$$

where for maximum simplicity we put

$$g(x) = 1$$

Since

$$\int_0^\infty dv\ v^{-l-1}\exp(izv) = \exp(-i\pi l/2)z^l\Gamma(-l)$$

$$F(l) = \int_0^1 dx\ x^l\exp(-i\pi l/2)\Gamma(-l)g(x)$$

So

$$(l + 1)F(l) = \exp(-i\pi l/2)\Gamma(-l)$$

So by eq.4

$$F(v) \sim i/v$$

Now suppose zero is not one of the limits in eq.5 but the integration runs from a to b

$$F(v) = \int_a^b dx\ g(x)\exp(ixv)$$

In eq.11 we can change the contour so that it avoids the origin in the complex x plane. In the new eq.8 the integrand remains finite and the range is finite so the integral is finite. In other words, $\hat{F}(l)$ has no poles in l. This means that F falls faster than any power.

Another way of seeing this is to integrate by parts. This method is fully explained in ref.2,p.47 where we find the answer

$$F(v) \sim \sum \exp(ixv)v^{-l-1}g^{(r)}(x)\Big]_a^b$$

Thus unless a or b is zero, F(v) has exponential behaviour for large v, but the direction v real is slightly exceptional.

If mathematics has its folklore, it also has its myths, presumably folklore distorted in the telling. In particular the idea seems to be prevalent that in eq.11 the dominant region of integration is x near zero. We have seen that on the contrary, the integral in eq.11 is dominated by end-point contributions.

Unfortunately, Mellin transforms are about as truthful as the witches of Macbeth [ref.3]. Here is an example to show one of the difficulties (and incidentally that some of the contributions are not from the end-points).

You are challenged to find the mistake in the following lines.

Let

$$F(v) = \iint dxdy\ g(x)h(y)\exp(ixyv)$$

$$\hat{F}(l) = \Gamma(-l)\exp(-i\pi l/2)\iint dxdy\ (xy)^{l}\, g(x)h(y)$$

$$= \Gamma(-l)\exp(-i\pi l/2)\left\{\int dx\ x^{l}g(x)\right\}\left\{\int dy\ y^{l}h(y)\right\}$$

Now as discussed above the x,y contours can be distorted to avoid the x and y origins respectively and it would appear again $\hat{F}(l)$ has no poles in l. This conclusion is false as is shown by the toy integral

$$\iint dxdy\ \exp(i(x^2 + y^2 + xyv)) = (2\pi/v)/\sqrt{(1-4v^{-2})}$$

which may be shown by completing the square arguments.

The practice of inventing easy examples to illustrate steam hammer techniques is very useful when trying to mend the steam hammer!

Let us dispose of two red herrings. Eq.7 is correct even for negative values of z, even though it is true that the imaginary part of z must be at least a little positive. $\Gamma(-l)$ has poles at l=0,1,2,... but the contour in eq.3 runs to the left of them.

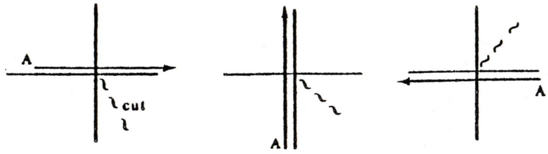
The resolution of this paradox is in the many-valuedness of one of the functions involved. How often do we think that they must have been invented to make life complicated when all the time the simplest functions are many-valued? For example

$$(xy)^{l} = x^{l}y^{l}$$

is true only up to factors of $\exp(2i\pi l)$.

Suppose that in trying to apply our treatment of eq.11 to eq.13 we distort the contours as follows. Let the y integration avoid the origin in the complex y plane by going through the point $i\epsilon$, and similarly for the x integration contour in the x plane. For fixed x, as y varies from $-\infty$ to $+\infty$ the product xy traces out the path illustrated in the figures.

Path 1 is for $x > 0$, path 2 for $x = i\epsilon$, path 3 for $x < 0$.



Now in eq. 14 there is the many valued function $(xy) \to (xy)^l$. Where should its cut be? For eq.7 the cut in $z \to z^l$ is in the lower half-plane so in fig.1 the cut is where shown. If eq.17 were right, then as x goes negative, the xy path would sweep the cut round so that the "A" end of the xy path would finish on the under side of the cut. However we really want to be on the upper side; we can see this by noting that in eq.13 as far as the argument of exp is concerned, x and y negative is the same as x and y positive and we expect this equivalence to be preserved in eq.14. A more detailed argument would bring into account that in eq.7, z should have a positive imaginary part.

Here we could replace z by $\lim(z+i\epsilon\ ;\ \epsilon \to 0)$. Hence the offending equation is eq.15. and it should be replaced by

$$\hat{F}(l) = \Gamma(-l)\exp(-i\pi l/2)\iint dxdy\ g(x)h(y) \times$$

$$x^l y^l (1 + \theta(-x)\theta(-y)(-1 + \exp(-2\pi i l)))$$

We still need a little manipulation and care over signs, but the term with the theta functions produces the right answer

$$F(v) \sim 2\sum r!(i)^r g_r h_r v^{-r-1}$$

where

$$g(x) = \sum g_r x^r \ ;\ h(x) = \sum h_r x^r$$

It would be impossible to explain all the subtlties of Mellin transforms, so I leave you to prove

$$\iint dxdy\ f(x,y)\exp(iv(x - y)) \sim v^{-2} f(0,0)$$

References.

There is no connected account of Mellin transforms but some information is given in

1a. R.Courant & D.Hilbert Methods of Math. Physics (1962)
1b. The Bateman Manuscript
2 . A.Erdelyi Asymptotic Expansions (1956)
3. W.Shakespeare Macbeth

Letters

Sir,

I refer to your October, 1973 issue, pages 22 & 49. The solution given to problem 8 b) seems questionable, if not wrong. It evidently assumes that Laura brakes steadily so as to cross the pad at 15 m.p.h. and reach the light at 0 m.p.h., exactly as it changes, but if she starts braking a little earlier, so as to reach the pad at 7.5 m.p.h., and then holds this speed, again reaching the light exactly as it changes, it will be found that she has (eventually) been delayed by 6.75 seconds instead of 8.

I conjecture that this strategy gives the least possible delay and I claim that this is a sensible, perhaps the most sensible, interpretation of the ambiguous condition "She always drives as fast as possible".

The point seems to be to be of real practical interest. Many non-mathematical motorists seem unaware of the advantages of similar strategies, which, by the way, save petrol as well as time.

G.H. Toulmin

Cheltenham

Sir,

With reference to Mr. Toulmin's letter, by the condition "She always drives as fast as possible", I intended to mean that at every instant, her speed was maximal subject to speed limits, acceleration conditions, and never passing a red light. I agree that this should have been worded explicitly.

However, while Mr. Toulmin's solution is interesting, he presupposes that the light will start to change as soon as Laura crosses the pad. Unfortunately, this is not always the case. All that can be said is that the sooner she reaches the pad, the sooner the light will change (in the weak sense). Mr. Toulmin has her arriving at the pad 15/16 of a second after she actually did: in that time, cross-traffic could have activated the pads on the side-road, causing a delay of perhaps 30 seconds before the lights went green for Laura. I admit that this is unlikely, but Laura adopts a minimax strategy.

N.H.G. Mitchell

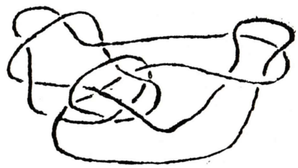
Trinity College

Problems Drive

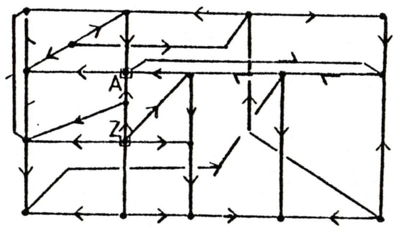
The problems in the 1974 Archimedeans versus Invariants problems drive were set by the 1973 winning pair, Colin Vout and Martin Brown.

The answers are on the inside back cover.

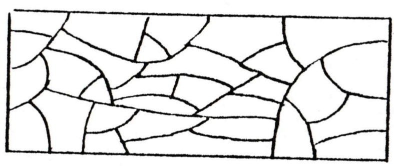
1)(a) Is this a knot or not?



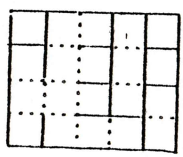
(b) Can you get from A to Z in this complex of one-way streets?



(c) Below is a supposed picture of a wall tile which has progressively cracked, that is, one new crack at a time. Each new crack has its ends in the middle of a previous crack or on the edge. Could this in fact have happened?



(d) Can the following ready-creased sheet of paper be folded up, with no 'tucking-in' needed? (Bold and dotted lines are oppositely creased.)



2) Let $x = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$ be the expression of the integer x in prime factors such that $p_i \neq p_j$ if $i = j$. (i) Let $d(x)$ be the number of divisors of x (including 1 and x). Express $d(x)$ in terms of the a's. (ii) Define $S(x) = 1$ if x is a perfect square, else 0. Express $S(x)$ in terms of $d(x)$. (iii) Let $N(x)$ be the number of representations of x as a sum of two perfect squares. Express $N(x)$ in terms of $S(x)$ and thus in terms of a sum involving only terms like $d(y)$.

3) A frame is defined as a networks of sticks, or finite line segments, drawn in the plane such that each stick crosses exactly three others, alternately 'over','under','over' or vice versa. Each intersection is to involve only two sticks, one under and one over.

Find and draw all possible frames with at most seven sticks.

4) Find particular solutions, valid in $R \setminus Z$, to the following differential equations, where $f^{(r)}(x) = D^r f(x)$ and $[a]$ is the least integer not more than a.

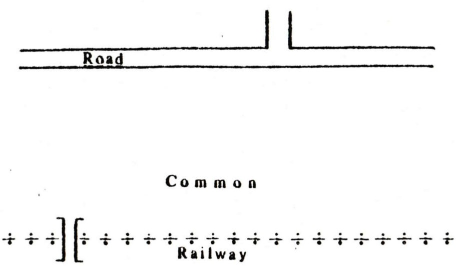
(i) $f^{[f^{[f(x)]}(x)]} = 1.$ (ii) $f^{[f^{[f^{[f(x)]}(x)]}(x)]} \equiv 1$

5) Mr C.T. Commuter, on his way to the underground station in the morning, has to cross Drowntwash Common. This involves crossing a railway bridge, walking over the common, crossing a main road and going down a side road. The traffic distribution is such that the probability of waiting t minutes to cross the main road is

$$p(\text{waiting less than } t \text{ minutes}) = (t + T)/2T$$

He always walks at the same speed and can cross the common dirctly in a minutes and then walk to the side road in b minutes.

If he chooses his routes to and from the station as sensibly as possible, does he take longer to cross the common on his way to or from the station? Also work out how long he takes to cross the common on the way to the station.



6) This problem is the subject of 'The Cups Problem' on p.24.

7) Solve these simultaneous equations in integers X,Y,U,V between 1 and 50.

$$X^2 + 5Y^2 = U^2$$
$$X^2 - 5Y^2 = V^2$$

8) Find all finite commutative groups such that the product of all the elements of the group is not the identity.

9) Our hero finds himself surrounded by four baddies, at the corners of a square with him at the centre. A wry smile plays across his lips as he assesses the situation; he knows that all four can run at the same speed, which, owing to their not having spent three hours a day training, is just three-quarters of his. But all of them, like him, have infinitely fast reactions.

Will our rugged hero escape? If so, how and why?

10) G is a finite abelian multiplicative group generated by x_1, x_2 which commute. Entries in this cross-group are elements of G, written with a power of x_1 or x_2 only in each cell.

Across. 1. $\sqrt{x_1}$ 3. $(x_1x_2)^n$ where n is the smallest positive integer such that the problem has a unique solution.

Down. 1. $\sqrt[5]{x_2}$ 2. $\sqrt[3]{x_1}$

1	2
3	

11) In this alphametric, each letter represents a digit in the base of ten. There are no leading zeroes.

What is the value of THISVISIT ?

```
  THEINVARIANTS
+  ARCHIMEDEANS
= PROBLEMSDRIVE
```

12) "We can find the probability that a number is prime by two different methods.

(i) The number of primes < n is denoted by P(n). It is well known that $P(n) \sim n/\log n$ as $n \to \infty$. Hence

$$p(n \text{ is prime}) \sim 1/\log n \text{ as } n \to \infty.$$

(ii) Obviously

$$p(n \text{ is prime}) = p(2 \nmid n \;\&\; \ldots)$$ where the ellipsis indicates all primes less than n. Since the events

$2 \mid n$, $3 \mid n$, ..., are independent,

$$p(n \text{ is prime}) = \prod p(p \nmid n) = \prod (1-1/p).$$

It can be shown (honestly!) that

$$\prod_{p<n} (1-1/p) \sim 2\exp(-g)/\log n$$

as $n \to \infty$, where g is Euler's constant, $g = 1.124...$.

Thus

$$p(n \text{ is prime}) \sim 1.124/\log n$$

This is a contradiction, and the end of mathematics."

What is wrong with the above argument?

13) Two cicuits of model car track are laid out, crossing at several points. Circuit A is of length 60, circuit B of length 35 and the crossing points relative to the cars starting point are at distances:

A	7	15	18	34	48	57
B	9	14	17	19	27	63

Do the cars crash, and if so where?

14) What is the next number in the following sequences?

(a) 3,3,5,4,4,3,5,5,4 ? (b) 1,2,720,?

(c) 2,1,13,19 97,211,? (d) 6,8,5,8,4,0,7,3,4,?

Exchanges

The following periodicals were exchanged with Eureka this year. They have been placed in the Scientific Periodicals Library.

Gazeta Mathematica seria A,B; Journal of the Mathematical Association of Ghana; Scientia Sinica; Studia Scientiarum Mathematicarum Hungarica; Analele Stiintifice ale Universitatii "Al. I. Cuza"; Annales Universitatio Scientarium Budapestensis de Rolando Eotvos Nominatae; Nordisk Matematiski Tidiskrift; Glasnik Matematicki; Revue Roumanie de Mathematiques Pures et Appliquees; Analele Universitatii din Timisoara; Science et Techniques; Abstracts of Bulgarian Scientific Literature.

Fluids & Geometry

by M.K. Jeeves

To my surprise the other day, I came across a connection between fluid dynamics and hyperbolic geometry. In this article, I propose to trace the somewhat tortuous path between the two.

Consider an open plane domain $D \subset C$, say bounded with p-smooth boundary σ. (This is too strong a condition but saves trouble). Let $\bar{D}$ be $D \cup \sigma$. Suppose that in this there is a solenoidal irrotational flow represented by f satisfying certain boundary conditions. Then we can represent the system by a Green's function, $G(z,t)$, where for any $t \in D$, $G(z,t)$ is harmonic for $z = t$ and $G(z,t) - \log|z - t|$ is harmonic for $z \in D$.

Having now disposed of the fluid, we can get down to some real mathematics. By Cauchy's integral formula, if f is analytic on D and continuous on $\bar{D}$,

$$2\pi i f(t) = \int_\sigma f(z)/(z - t)\, dz$$

thus

$$2G(z,t) = -(\log|z - t| + \log\overline{|z - t|}) + h(z,t) \qquad 1$$

where h is harmonic in z on D. Differentiating wrt t,

$$1/(z-t) = -2G_t(z,t) + 2h_t(z,t)$$

since G is zero on σ by boundary conditions,

$$-\pi i f(t) = \int_\sigma f(z) h_t(z,t)\, dz$$

by Stokes' theorem,

$$(\pi/2) f(t) = -\iint (f h_t)_{\bar{z}}\, dxdy$$

where $f_z = (f_x - i f_y)/2$, $f_{\bar{z}} = (f_x + i f_y)/2$ $\quad z = x + iy$

So if $K(z,t) = -2G_{z\bar{t}}(z,t)/\pi$, we have

$$f(t) = \iint f(z) K(z,t)\, dxdy \qquad 2$$

Here K is called the kernel function for D. We can put this property (2) of K in another light. The analytic functions on D form a C-vector space $L_2(D)$ with an inner product

$$\langle f,g \rangle = \iint f\bar{g}\, dxdy$$

and norm

$$|f| = \sqrt{\langle f,f\rangle}$$

Then $L_2(D)$ is a Hilbert space, that is, it is complete under the metric

$$d(f,g) = |f - g|$$

It is now easy to show that there is an orthonormal basis $e_1, e_2, e_3, \ldots$ such that any f can be approximated arbitrarily closely by a finite linear combination of the e's.

Consider now the dual space $L_2(D)^o$ of bounded linear functionals from $L_2(D)$ to C. There is an interesting

Theorem.

Any bounded functional $l \in L_2(D)^o$ can be written as

$$l(f) = \langle E,f\rangle$$

for some unique $E \in L_2(D)$.

We shall not prove this theorem but instead apply it to the functional l given by $l(f) = f(t)$. This is bounded as $\bar{D}$ is compact so there is a unique K analytic in z such that

$$f(t) = \langle K,t\rangle$$

which is just (2).

It is also easily seen that the kernel function possesses an expansion in terms of an orthonormal basis

$$K(z,t) = \sum e_i(z)\overline{e_i(t)}$$

(cf the similar result for the Green's function) and so K is anti-analytic in t.

We can use this kernel function to establish the surprising

Riemann mapping theorem.

Every bounded domain $D \subset C$ can be mapped conformally 1-1 onto the unit disc so that for given $t \in D$ and $\theta \in R$,

$$f(t) = 0 \quad , \quad \arg f'(t) = \theta$$

We can put

$$f = \int_t^z K(w,t)dw.\exp(i\theta).\sqrt{(\pi/K(t,t))}$$

Obviously any two domains can be mapped conformally to each other (through the unit disc). Let $f:D \to E$ be such a map. Then if $(e_i(z))$ is an orthonormal basis for $L_2(D)$, $(e_i(z)f'(z))$ is one for $L_2(E)$, and so

$$K_D(z,t) = K_E(f(z),f(t))f'(z)\overline{f'(t)}$$

Thus if $w = f(z)$, $z \in D$, $w \in E$, we have

$$K_D(z,z)|dz|^2 = K_E(w,w)|dw|^2$$

so that $ds = \sqrt{(K_D(z,z))}|dz|$ defines a conformally invariant length for D by putting

$$\text{length}(\gamma) = \int_\gamma \sqrt{(K_D(z,z))}|dz|$$

In fact, this length defines a Riemannian metric for D.

Let us compute this for the unit disc. Clearly

$$f(z) = \exp(i\theta)(z - t)/(1 - \bar{t}z)$$

is a conformal map satisying the criteria of the Riemann mapping theorem , which suggests, as is indeed the case, that

$$K(z,t) = 1/\pi(1-z\bar{t})^2$$

so that the conformally invariant length is given by

$$ds = |dz|/(1-|z|^2)^{1/2}$$

which is just the well-known Poincare model for the disc.

Thus we can derive the hyperbolic geometry of the disc as an intrinsic property of conformal mappings, not merely an ad hoc supposition. I hope this gives some indication of the flavour of Riemannian geometry and its connections with other branches of mathematics.

Noughts & Crosses

by S.J. Taylor

1. Introduction

Many games players write off all Noughts and Crosses games because every 3*3 game that they have played since primary school (or earlier!) has ended in a draw. Perhaps they have even tried and mastered the trivial 3*3*3 game (exercise: prove that the first player wins this game on his fourth move at the latest). However the 4*4*4 game is rich in possibilities and I shall try and illustrate some of these. Some games manufacturers have recognised its potential and sell plastic 3-D boards for around £1, however pencil and paper will suffice if you don't mind stacking boards in your head.

Before looking at sophisticated strategies, a few remarks on elementary tactics.

```
figure 1a          X  X  X  .


       1b          C  X  X  .A

                      X

                         X

                            .B
```

The crudest possible strategy is to line up 3 counters and hope that your opponent overlooks the threat (fig. 1a): this will suffice for beating grandma but otherwise is a waste of time. The next try is to simultaneously establish 2 lines of 3 counters as in fig. 1b. Opponent takes A or B and you win by taking the other. Alas - to achieve this you must take the pivot square C on your previous move and if the opposition is wise to this trick you'll be perpetually frustrated. Let us call the configurations 1a, 1b, threat and double-threat respectively. Then the amateur's strategy will be to manoeuvre so that he can play a series of threats forcing his opponent's replies, and finish him off with a double-threat. Inspecting the geometry of the 4*4*4 board we note that 7 lines pass through 16 squares and only 4 lines through the remaining 48; call these privileged 16 squares, the centre. It is then natural to augment our crude strategy by attempting to take as much of the centre as possible. This will lead to an enjoyable game comparable to chess without openings and positional play.

Before our beloved TITAN computer passed away it was programmed to follow this simple strategy. It analysed all the threats and double-threats to a depth of 3 moves for each side, and when it could find no obviously best move using this analysis chose a move so as to increase its chances of forcing a double-threat later. Thus it showed a natural preference for central squares. The program played 18 games before TITAN was scrapped, winning 9 drawing one and losing the remainder. Most of the losses were due to the programmer demonstrating this algorithm's weakness against sophisticated strategies. Among the program's scalps are 2 other computer programs and the secretary of the Archimedeans (twice).

2. Some sophisticated strategies.

(a) Using planes.

The 4*4*4 board contains 18 4*4 planes (including 6 diagonal planes). It is much simpler to analyse the play on a plane than on the whole cube. For example, consider the situation in figure 2:

figure 2.

```
+---------+
- X . . X -
- . . . . -
- . . . . -
- X . . . -
+---------+
```

Suppose the player of the crosses has the move and that he can play on this plane without considering the rest of the board (which is often the case). Then he can win as follows;

```
+---------+    +---------+    +---------+
- X . . X -    - X . . X -    - X . . X -
- . . . . -    - O . . . -    - O . O . -
- . . . . -    - X . . . -    - X X . . -
- X . . . -    - X . . . -    - X . . . -
+---------+    +---------+    +---------+

+---------+    +---------+    +---------+
- X X O X -    - X X O X -    - X X O X -
- O . O . -    - O X O . -    - O X O . -
- X X . . -    - X X . . -    - X X X . -
- X . . . -    - X O . . -    - X O . . -
+---------+    +---------+    +---------+
```

We would like to know when it is possible to overwhelm the opposition in this fashion. Clearly we will need at least 3 counters on the plane. Further as soon as the opposition gets a counter on our plane, it is very hard to overwhelm him. Thus the most important case for analysis is that of the 3-0 lead. I have completely solved this problem and consider the result so pretty that I shall call it a theorem

Definition: the squares * in figure 3 are diagonal squares (ds),
the remainder are non-diagonal squares (nds).

figure 3.

```
+---------+
- * . . * -
- . * * . -
- . * * . -
- * . . * -
+---------+
```

Theorem: Leading 3-0 on a plane you can/cannot overwhelm your opponent as follows:

diagonal squares

3	Always possible.
2	Possible if and only if, the nds is collinear with at least one ds.
1	Possible if and only if, a chess knight(!) may jump from one nds to the other, and the ds is collinear with a nds.
0	Never possible.

Regrettably it seems that this theorem must be proved by exhaustively checking all the independent cases. Can anyone find a more aesthetically satisfying proof? Having ascertained that one has a winning 3-0 lead one needs to know how to achieve the win. The following are helpful:

1: The winning process is always a sequence of 4 threats, leading to a forced double-threat.

2: Leading 3-0 (in a winning position) start thus:

diagonal squares

3	Any threat will suffice.
1 or 2	Find a ds and nds which are collinear. Take the remaining ds in this row (forcing a nds reply).

These results will enrich anyone's play. Since there are 18 planes it is difficult for the players to keep track of the situations on all the planes and to decide which planes to attack and which to defend. However if only one of the players understands the tactics of planes then the informed player invariably wins and without too much trouble. Finally a warning: whilst you are busy on your plane executing the winning series of threats you may establish a threat for your opponent elsewhere on the board. This will wreck all your work! A little planning usually deals with this.

(B) systematic opening play.

It is convenient to define what I shall call the 2 sub-centres.

```
+---------+   +---------+   +---------+   +---------+
- O . . O -   - . . . . -   - . . . . -   - O . . O -
- . . . . -   - . I I . -   - . I I . -   - . . . . -
- . . . . -   - . I I . -   - . I I . -   - . . . . -
- O . . O -   - . . . . -   - . . . . -   - O . . O -
+---------+   +---------+   +---------+   +---------+
```

Let us call the 8 squares I, the inner sub-centre, and the 8 squares O, the outer sub-centre.

It is natural to study 'openings' leading to the quick destruction of the opposition on the planes. What follows is of great use when you have the first move. An acquaintance from St. John's suggested to me the following opening strategy: take a suitably selected 4 of the 8 squares of a sub-centre; then if your opponent takes the other 4 squares of the selected sub-centre, he loses immediately. Player 1 (who starts the game), playing X, can always force fig. 4a or fig. 4b assuming the opponent plays in his sub-centre. (This may easily be verified by the reader)

fig. 4a.

```
+---------+   +---------+   +---------+   +---------+
- A . C . -   - . . . . -   - . . . . -   - . . . . -
- . B . D -   - . X X . -   - . X O . -   - D'B'. . -
- . . . . -   - . O O . -   - . O X . -   - . . . . -
- . . . . -   - . . . . -   - . . . . -   - . .C'A'-
+---------+   +---------+   +---------+   +---------+
```

fig. 4b.

```
+---------+   +---------+   +---------+   +---------+
- . . . . -   - . . . . -   - . . . . -   - C'. . . -
- . A . B -   - . X X . -   - . X O . -   - B'A'. . -
- . . . . -   - . O X . -   - . O O . -   - . . . . -
- . . . C -   - . . . . -   - . . . . -   - . . . . -
+---------+   +---------+   +---------+   +---------+
```

From each of figs. 4a and 4b, the first player wins. in 4a, he plays the threats A B,C,D forcing A',B',C',D' whilst in 4b, he plays A,B,C forcing A',B',C'. Then player 1 overwhelms his opponent with his lead on the lowest plane. Thus it is suicide for player 2 to contest a sub-centre with player 1. However player 2 must play some counters in the chosen sub-centre since player 1 will soon achieve several 3-0 leads on the planes passing through his sub-centre. So player 2 is in a dilemma against systematic opening play. No defence is known to this strategy of controlling a sub-centre. There are however too many possibilities to prove that player 1 has a forced win.

It is clear that with best play the 4*4*4 game is a win for the first player or a draw. It seems that the first possibility is the true situation; perhaps someone with a lot of spare time can prove this.

I hope this article illustrates that there is still some life in noughts and crosses. I trust readers will play a few 4*4*4 games with their friends. Providing that you understand this article and your opponents don't read Eureka you should win the first few games. Good luck!

And Now....

by Nigel Black

Once upon a space-time, there was an aging kernel who lived in the middle of an ancient forest. He led a stable existence, living off an income left by his uncle, a well known cardinal, not large but allowing him a certain degree of freedom. Few people came within his orbit.

One day I was randomly walking across the fields when I chanced upon this ideal ring of trees. Two lovers sat entranced beneath the shining sun. Suddenly, I looked away and saw the colonel approaching.

"Hello", I nervously projected. I tried to gauge his reaction.

"What a degenerate case!" he thundered, "it's rank bad taste and I will try and nullify it."

"Oh come on", I replied, "all couples have their moments. it's merely a normal extension of the transcendental element between them - it only goes skin deep."

The atmosphere grew tensor. I noticed a sudden change in the colonel.

"Metamorphism", he said.

"Yer what?" I replied.

"I said I met a Morphism, you know, the Morphisms that came from Kennington, (of course I don't hold much with integration you know), who live down the road in their homomorphism. Yes, I was out walking the other day when up drew their automorphism and stopped beside me. The window wound down and out of it appeared this head - " I'ze-a-morphism man", he said. "I know, you know", I replied, "but why are you telling me this?'. "Well sir, I'm not an 'epimorphism, not 'appy at all, no sir. It's the cost of living. I used to be a commuter in the centre of London, but I felt I was losing my identity. Now there are so many constraints that I can do no work at all. And then there's the family Susan Morphism, Polly Morphism, and my cousins who have extracted their roots from north London - the 'endon Morphisms - there's a whole compact space, not to mention Juliet Morphism and Romeo Morphism.

"Look my deer chap, I don't know why I'm telling you all this", continued the colonel, "come home and have some tea with me".

We set off into the forest.

"It's affine day", I noted, "but why have these trees got square roots?".

"Oh, it makes them easier to extract, and provides a better base for the logs, you know", replied the colonel.

"Economical optimization I suppose, but they're still being cut into foot long sections".

"All in its proper time, you know. Now that the Relativities board have reported we can't do everything simultaneously. Too much of this decimal lark is a bad thing. You know, when they launch the next module, it'll probably be travelling in a metric space!".

Snaedemihcra?

by Charles Bogle

A certain society was holding an election for President. The voting was done by the 5 remaining committee members, the Caterer, Junior Treasurer, Minutes Secretary, Secretary and Vice President. These posts were occupied by Straw, Gengis, Less, Steel and Tyger but not necessarily in that order.

Each committee member was up for election, and nobody could vote for himself. Thus in order to be elected, it was necessary to solicit votes from the other candidates. Before the election, each candidate had received the promise of just one vote.

When the ballot papers were examined, it was discovered that although each candidate had received exactly one vote, nobody had voted for the person they had promised to.

It was seen, however, that

1. Less had voted for the Junior Treasurer.
2. Less was the only person to vote for the candidate whom the person he had promised his vote to had promised his vote to.
3. Tyger had voted for the Secretary, who had promised to vote for Tyger.
4. Gengis had voted for the Minutes Secretary, who had been promised Straw's vote.
5. Steel had voted for the Vice President.

Who was the Caterer?

Any resemblences to any society or persons living or dead is entirely intentional.

The answer is on p.28.

Functional Equations

by K.J. Falconer

Given some function, f, mapping a domain D into itself, it is usually a straightforward matter to calculate expressions of the form $f(f(x))$, $f(f(f(x)))$ and so on. In this article we look at the converse of this process, namely, given some function $g(x)$, we ask whether an f exists such that $f(f(x)) = g(x)$, and if so what properties are forced on f (is it possible for f to be continuous? and so on). For example, as we shall show, letting $g(x)$ be $x^2 + 2$, $-x$, $x^2 - 2$ we have infinitely many continuous solutions, infinitely many solutions all discontinuous, and no solutions respectively.

In what follows, the notation defined by $f_1(x) = f(x)$, and $f_{n+1}(x) = f(f_n(x))$ $(n > 1)$ is adopted, so that the problem mentioned above amounts to finding solutions of $f_2(x) = g(x)$. In fact, we shall consider the slightly more general problem of solving $f_n(x) = g(x)$ where f and g are to map D to itself; though for our purposes D may be taken to be the real or complex numbers. Finding necessary and sufficient conditions for $f_n(x) = g(x)$ to have solutions is a complicated problem involving homomorphisms of infinite graphs. Consequently, we shall in what follows derive necessary conditions for solutions only.

Firstly we note that any solution $f(x)$ of

$$f_n(x) = g(x) \qquad x \in D \qquad 1$$

must commute with the given function $g(x)$. For (1) gives

$$f(g(x)) = f(f_n(x)) = f_n(f(x)) = g(f(x))$$

In particular this gives

$$f(g_r(x)) = g_r(f(x)) \qquad x \in D$$

so that considering the fixed points of g_r we get that if y is a fixed point of g_r, then

$$f(y) = f(g_r(y)) = g_r(f(y))$$

that is $f(y)$ must also be a fixed point of g_r. Thus if we let K_r be the set of y such that $g_r(y) = y$; then any solution f of (1) must map K_r into itself. In fact, a solution f must map K_r onto itself, for if $g_r(y) = y$ then $y = f(f_{nr-1}(y))$ and $g_r(f_{nr-1}(y)) = f_{nr-1}(y)$ so that y is the image under f of some element of K . It is also clear that f acting on K is 1–1; for if $f(y) = f(z)$, $y, z \in K$, then $f_{nr}(y) = f_{nr}(z)$ so $g_r(y) = g_r(z)$ giving $y = z$. Thus we have shown that any solution f of (1) must be a permutation on each K.

We can now define sets P_r by

$$P_r = \{x \in D : g_r(x) = x;\ g_s(x) \neq x,\ 0 < s < r\}$$

$$= K_r \setminus (K_1 \cup K_2 \cup \ldots \cup K_{r-1})$$

If r is the least positive integer such that $g_r(x) = x$ (if any such r exists) then we say that x has period r wrt g. Hence P is the set of elements of D with period just r wrt g. As any solution f is a permututation on each K_r, it follows that any such f must be a permutation on each of the P_r. We now use this fact to place restrictions on the number of elements in any P_r if solutions to (1) exist. Suppose $y \in P_r$ then $f_{nr}(y) = y$; $f_{nl}(y) \neq y$ if $0 < l < r$. Hence if y has period p wrt f, then $p \mid nr$; $p \nmid nl$, $0 < l < r$. Thus p is a multiple of r, for otherwise $p = ms$ with $m \mid n$, $s \mid r$ and $s < r$, so that $p \mid ns$, which is a contradiction.

So let $p = rn_1 \ldots n_k$ where $n = n_1 \ldots n_t$ (n prime, $t > k$) then $rn_1 \ldots n_k \nmid n_1 \ldots n_t l$ if $l < r$

or $$r \nmid n_{k+1} \ldots n_t l \qquad \text{if } l < r$$

This implies that $n_i \nmid r$ if $i = k+1, \ldots, t$; for if $n_i h = r$, say, with $i > k$, we get $r \mid n_{k+1} \ldots n_t h$ with $h < r$ which contradicts (2). Thus $n_{k+1}, \ldots, n_t$ do not divide r and are all prime, so that $p = rn_1 \ldots n_k$ where the $n_1, \ldots, n_k$ include all prime factors of n which divide r. Thus if we write the product of the prime factors of n which divide r as (n, r^*), we have shown that p must be a multiple of $r(n, r^*)$. (In fact, $(n, r^*) = \max\{(n, r^q) : q = 1, 2, 3, \ldots\}$)

Now, f permutes the elments of P_r, so if P_r is finite, f will partition P_r into disjoint cycles of the form $(y, f(y), \ldots, f_q(y))$ and we have shown that the length p of each cycle must be a multiple of $r(n, r^*)$ elements.

Hence we have proved that a necessary condition for (1) to have a solution in D is that $r(n, r^*) \mid |P_r|$ all r.

We can now go back to one of our original examples,

$$f(f(x)) = x^2 - 2 \qquad x \in R \qquad (3)$$

In this case, as may easily be seen, $P_1 = \{-1, 2\}$ and $P_2 = \{(-1 \pm \sqrt{5})/2\}$. Our condition requires that $2(2, 2^*) \mid |P_2|$, that is $4 \mid 2$. So it follows that (3) has no solution. Indeed, if we apply this criterion to the general quadratic case, we find that

$$f(f(x)) = ax^2 + bx + c \qquad x \in R, a \neq 0$$

has no solution if $(1+b)(b-3) < 4ac$. This is obtained just by considering $|P_2|$, stronger conditions may be obtained by considering $|P_r|$ for larger values of r.

We conclude by looking at one of the other examples given

$$f(f(x)) = -x \qquad x \in R$$

to see that no continuous solutions exist, and how discontinuous solutions may be constructed. In this case, $P_1 = \{0\}$ and $P_2 = R \setminus \{0\}$ which immediately require f to be injective with $f(0) = 0$; also commutativity of f and "-" gives $f(-x) = -f(x)$ so that f must be an odd function. So suppose that $f(1) = y = 0$. Then $f(y) = -1$; $f(-1) = -y$ and $f(-y) = 1$. Hence, if $y > 0$, $f(1) > 0$, $f(-y) > 0$ and if $y < 0$, then $f(y) < 0$, $f(-1) > 0$. So as $f(0) = 0$, f cannot be monotonic. But all continuous 1-1 functions on R are monotonic, so no solution of (4) is continuous.

A solution of (4) may be constructed by dividing the positive reals into two sets of equal cardinality and setting up a bijection between them. For example, let $A = (0,1] \cup (2,3] \quad \ldots$; let $B = (1,2] \cup (3,4] \quad \ldots$, and let $T:A \leftrightarrow B$ be the map $T:x \rightarrow x+1$. Then we can define f as follows

$$
\begin{aligned}
&f(0) = 0 \\
&f(x) = 1 + x; \; f(-x) = -1 - x \qquad \text{if } x \in A, \\
&f(x) = 1 - x; \; f(-x) = -1 + x \qquad \text{if } x \in B.
\end{aligned}
$$

It is easily seen that f satisfies (4). Constructions of a similar, but more complicated, nature may be used in other cases where solutions exist.

Recursion

(Found on a card in room A, the Arts School)

Overheard at the Archimedean Bathday Party.

1. Hello, I'm John. 2. Oh, I'm - er - Mary.
3. D'you do maths then? 4. Yes, do you?
5. Yes, it's ggrreat! 6. I have sugar frosties too.
7. Ggrreat! 8. $x^4 + y^4 = -1$
9. Ggrreat! 10. Am I boring?
11. Yes, ggrreat! 12. Oh, let's change the subject.
13. Ggrreat! 14. What d'you think of the political situation?
15. Ggrreat! 16. What's your name again?

<Go to step 1>

ACKNOWLEDGEMENTS

I would like to thank the other members of the staff, Terry Lyons who helped with the computing, and Geoff Chapman who looked after the accounts. I would also like to thank our contributers, and to acknowledge the assistance of the university computing service. The congratulations are due to them, the brickbats I take upon myself.

RGEP

The Cups Problem

by M. Brown

Question.

If M cups originally upright are inverted N at a time, so that eventually some that have been turned over will be turned upright again,

(a) when is it not possible to have all M cups upside down at some stage?

(b) when it is possible, estimate the minimum number of turns required to carry out the procedure.

Solution.

Let the number of cups unturned at the r'th turn be M_r, $r = 0, 1\ 2, \ldots$. Then the number unturned is $M - M_r$. Let x_r be the number of inverted cups which are turned upright at the r'th move, then $N - x_r$ of the unturned cups are turned upside down, hence

$$M_{r+1} = M_r - N + 2x_r \qquad 1$$

$$\Rightarrow \qquad M_r = M - rN + 2\sum_1^r x_q \quad ; \qquad r > 0 \qquad 2$$

Hence the problem is soluble if there is an r such that

$$0 = M - RN + 2\sum x_q \qquad 3$$

$$\Rightarrow \qquad M = RN \qquad (\text{mod } 2) \qquad 4$$

Hence there is no solution if M is odd and N even, and we shall prove that in all other cases there is a solution. This answers (a).

At each turn we have

$$0 < x_r < N \qquad 5$$

$$M_r - N + x_{r+1} > 0 \qquad 6$$

$$M - M_r - x_{r+1} > 0 \qquad 7$$

$$\Rightarrow \qquad (r - 1)N \geq x_r + 2\sum_1^{r-1} x_q \geq rN - M \qquad 8$$

$$x_1 = 0 \ , \ x_R = 0 \qquad 9$$

The problem is to find a minimum R for (3),(5),(8),(9) to hold for each $r \leq R$. Let $M = zN + h$, $z, h \in Z$ and $0 \leq h < N$.

Clearly if $N \mid M$ then the minimal number of turns is M/N; thus we can suppose that $h > 0$.

Eq.3 yields

$$0 = (z - R)N + h + 2\sum_1^R x_q \qquad 10$$

and hence we have

$$R > z = [M/N] \qquad 11$$

(where [a] denotes the greatest integer < a).

If z is sufficiently large, then it is clear that for a solution to exist, we choose x_r at each move to cancel h.

Suppose now that N and h are even; thus N and M are even. If we choose $x_2 = (N - h)/2$ then (8) holds for r = 2 if z > 1 but if we choose $x_3 = x_4 = \ldots = 0$, we see (8) holds for all r < z + 2 and for some R, (3),(5),(8),(9) hold; in fact we choose R = z + 1. This is obviously the minimum number of moves, from (11).

Suppose now that N is odd, M is even and z > 1. Then if z is even, h is even and we choose $x_2 = N - h/2$, with $x_3 = x_4 = \ldots = 0$. This is a solution of (3),(5),(8),(9) for R = z + 2. From (4),(11) we have that this is the minmum number of moves.

If now z is odd, then h is odd and we can choose as before $x_2 = n - h/2$, $x_3 = x_4 = \ldots = 0$ and get a solution in z + 1 moves, which is manifestly minimal.

The remaining case for z > 1 is n odd and m odd.

If z is even then h is odd and there is a solution in z + 1 moves with $x_2 = (N - h)/2$, etc.

If z is odd then h is even and there is a solution in z + 2 moves with $x_2 = N - h/2$, etc.

Hence for z > 1, we collect these in the formula

$$R = [M/N] + (1 + (-1)^N)/2 + (1 - (-1)^N)(3 + (-1)^{M+[M/N]})/4 \qquad 12$$

We now consider z = 1.

Lemma.

If M and N are both even or both odd and [M/N] = 1, N ∤ M, then there is a solution in 3 moves and any solution in 3 moves with [M/N] = 1 must have M ≡ N (mod 2), and for no such M,N is there a minimal solution in 2r + 1 moves, r > 1.

Proof.

If there is a solution in 3 moves,

$$0 = M - 3N + 2(x_1 + x_2 + x_3) \qquad 13$$

but $x_1 = x_3 = 0$, so

$$x_2 = (3N - M)/2 \qquad 14$$

=> M,N are both even or both odd.

Conversely, if M ≡ N (mod 2) and [M/N] = 1, then (3N - M)/2 is an integer and

$$0 \leq (3N - M)/2 \leq N$$

so we take

$x_2 = (3N - M)/2$ and we have a 3-move solution. Suppose there were a minimal solution in $2r + 1$ moves, $r > 1$, for some M,N with $[M/N] = 1$. Then, by (4), $M = N \pmod 2$ and so there is a 3-move solution, which is a contradiction.

Q.E.D.

We may now suppose N odd, M even. Suppose that there is a solution in an even number of moves, n for M,N. Consider the complementary position with M − N cups turned over on each move, and M cups altogether. Then it is clear that the n-move solution for M N also furnishes an n-move solution for M,M N and after n moves in the M,M–N problem the cups will all be the same way up.

Let m be a minimal solution for M,M–N and suppose there is a minimal solution for M,N in $n < m$ moves. If n is odd then from the lemma, $n = 3$ so $N = M \pmod 2$, a contradiction so n is even and so there is a solution for M,M − N in n moves, another contradiction. Obviously $m < n$ since m is even from (4). So the minimal number of moves for M,M–N equals the minimal number for M,N if $[M/N] = 1$, when M is even and N odd.

3 Problems

by Denis Thicket

(1) Define sequences s_i $(i = 0,1,2...)$ of 0's and 1's as follows: $s_0 = 0$, $s_1 = 1$, and for $i > 2$. $s_i = s_{i-1}$ followed by s_{i-2}. Let s_∞ be the sequence 10110101... of which each s_i $(i \neq 0)$ is an initial segment. Give a rule for finding the n'th term of s_∞ without writing the series out, and generalize.

(2) Prove by elementary means: if q is the least prime factor of the order of a finite group G then any subgroup of index q in G is normal.

(3) Prove or disprove the anti-Fermat conjecture: if N,a,b,c are positive integers such that

$$a^{N^{-1}} + b^{N^{-1}} = c^{N^{-1}}$$

then there are positive integers t,x,y such that

$$a = tx^N, \quad b = ty^N, \quad c = t(x + y)^N.$$

How to toss a coin

by A. Smith

Before you answer that, perhaps I should point out that the problem is, as it stands, rather meaningless. Mathematicians at Oxford have not been idle recently, however, and we have now discovered a context in which the question becomes a perfectly sensible one.

The story goes as follows. An undergraduate possesses a coin, which has probability p of showing heads when tossed. He invites a friend to guess the value of p, and to make the game more interesting, he suggests a gamble along the following lines. If the friend guesses the value g, then he is to receive a sum equal to £$[c - k(p - g)^2]$ if the latter is positive, but if negative he pays the owner of the coin a corresponding amount. (Assume that the values of c and k are such as to make the gamble interesting). The friend hesitates for a moment and then asks whether he is allowed to toss the coin a few time "just to get the feel of it". (You see, the friend knows his laws of large numbers and reasons that, if he can toss it enough times, the proportion of heads that will serve as his guess g is very likely to be close to p - the smaller $(g - p)^2$, the bigger his winnings). The owner replies that he can chhoose to make any fixed number of tosses, but it will cost him £c per toss, plus £d (for luck) every time a head appears.

Question: how many tosses should the friend choose to make? Should he in fact play the game at all?

We first notice that if he were to make n tosses, r of which turned out to be heads, and then were to guess g for p, then his loss, with respect to guessing g without any tosses would be given by $k(p - g)^2 + cn + dr$. But what should g be? Clearly it should depend on his beliefs about p posterior to performing the tosses; these in turn, depend on his beliefs prior to performing the tosses. Let us suppose the latter to be represented by $\pi(p) = Lp^{a-1}(1 - p)^{b-1}$, where

$$1/L = \int_0^1 p^{a-1}(1 - p)^{b-1}dp = \Gamma(a)\Gamma(b)/\Gamma(a + b)$$

and

$$\int_I \pi(p)\, dp \Big/ \int_{\bar{I}} \pi(p)\, dp$$

is the odds the friend would give on p belonging to the interval I rather then its complement $\bar{I}$ (it is through the latter kind of consideration that one chooses the values a and b).

Bayes theorem now provides us with the required posterior expression of belief;

$$\pi(p \mid n,r) \propto p(r \mid n,p)\pi(p) \propto p^{r+a-1}(1-p)^{n-r+b-1}$$

since

$$p(r \mid n,p) = {}^{n}C_{r}\,p^{r}(1-p)^{n-r}$$

By comparison with
$\pi(p)$, we see that the constant of proportionality is equal to

$$\Gamma(n + a + b)/\Gamma(r + a)\Gamma(n - r - b)$$

With respect to this posterior belief, the expected loss is given by

$$k\int_0^1 (p - g)^2 \pi(p \mid n,r)dp + cn + dr$$

So g should clearly be chosen to minmize this and after some elementary calculation, we obtain

$$g = (r + a)/(n + a + b)$$

Substituting this value, and carrying out the integration we find that $L(n,r)$, the minimum expected loss for given n and r, is of the form

$$L(n,r) = k(r+a)(n-r+b)/n+a+b)(n+a+b+1) + cn + dr$$

We do not, of course, know the value of r, for given n, but we can calculate $L(n)$, the expected loss for given n. This is defined by

$$L(n) = \sum_0^n L(n,r).p(r \mid n)$$

where

$$p(r \mid n) = \int_0^1 p(r \mid n,p)\pi(p)dp$$

After further calculations, we obtain

$$L(n) = kab/(a+b)(a+b+1)(a+b+n) + n(c + ad/(a+b))$$

It is now straightforward to find the optimal value n^* of n that minmizes the expected loss. If $L(n^*) > c$, the friend should politely tell the owner what to do with his coin.

SNAEDEMIHCRA!

The caterer was Gengis.

Book Reviews

Introduction to Measure and Integration
S.J. Taylor
CUP (1973) : . 1.90

This volume consists of the first nine chapters of Kingman & Taylor's Measure and Probability (CUP, 1966). It is a sound introduction to the courses on Lebesgue Integration and Measure Theory and would form a useful preliminary to that on Probability Theory. It covers the material in a similar way, extending the definition of measure to progressively wider classes of sets and defining an integral for wider classes of functions.

The treatment is not novel, but the last two chapters on linear functionals and special spaces cover some interesting material which is not in the schedules.

CB

Elliptic Functions and Elliptic Curves
P. Du Val
CUP (1973) : . 3.30

In this book, the author has attempted to link a discussion of elliptic functions to a survey of the properties of elliptic curves.

The sections on elliptic functions treat them in an essentially geometric way, by stressing the properties of the lattice as the basic concept. In these sections I found little, perhaps too little, analysis. The survey of curves assumes some knowledge of algebraic geometry but seems somewhat old-fashioned.

Although this might be of interest as a basic treatment, I do not feel it takes the reader deeply enough into some aspects of the subject.

MKJ

Introducing Real analysis
D.H. Fowler
Transworld (1973) : 80p

In this little volume, there is a very good introduction to the Analysis I course, suitable for reading before coming up . The author covers first the construction of the real line, with a well-motivated discussion of the upper-bound postulate. The subsequent chapters are devoted to properties of continuous and differentiable functions, and Taylor's theorem. There is an interesting final chapter on further development.

The style is discursive, even chatty, so this certainly does not form a reference work, but is probably worth the money to just read through as an introduction.

RGEP

Linear Programming
K. Trustrum
RKP (1971) : 40p

Linear programming is a subject which is easy to provide an elementary introduction to, but more difficult to establish at all rigourously. This book avoids some of the temptations and provides a logical, and well motivated treatment based on the simultaneous solution of the program and its dual.

There is a preliminary section on convexity, and chapters on the transportation and simplex algorithms, with an elementary final chapter on game theory.

This is perhaps one of the few books in the library of mathematics series to deserve some of the praise on its back cover.

CB

Theoria Modulelor
A. Solian
Editura Academici Republicii Socialiste Roumania (1973)

This extensive work provides a treatment of module theory from a category theoretic point of view. It would make a good reference although probably too turgid to read as an introduction. For anyone with a different backing in modules, this could be enlightening.

It has the disadvantage of being published in Roumanian, so that a glossary.is required for some of the technical terms.

MKJ

Talmudic Logic

by W. Felder

Jewish law is based in the Talmud, the collection of writings which form the basic authority along with the Holy Scripture. The interpretation and application of this has long been a complex legal, logical and theological problem. It is interesting to notice, therefore, that certain aspects of Talmudic study in operation for over five centuries, have a very strong relationship to modern systems of mathematical logic.

Let us state one of the problems of Talmudic thought. The statements of Scripture with legal relevance form a system S of propositions, of the general form $p(a)$, where p is a predicate and a some particular object. (we shall use p,q,... for predicates, a,b,c,... for objects and x,y,... for variables). We want to find a system T of propositions which includes the statements of S as theorems using a simple set of axioms and rules of inference, so that T will hopefully include a large range of other propositions as well, of relevance to matters not dealt with in ancient times.

Clearly one solution is the trivial system with no axioms, and as many rules of inference as are necessary to derive S, or any other set of propositions. We hope to improve this. We shall certainly make use of the propositional calculus with modus ponens ,that is ,

$$p \ , \ p \Rightarrow q \vdash q$$

which we read as "if 'p' and 'p => q' are theorems, so is 'q'". The interest of Talmudic logic is the introduction of other rules of inference to formalise such ideas as analogy, etc..

We shall consider in this article mainly kal vahomer, a rule depending on an ordering of predicates. Two predicates might be comparable in terms of a fine payable, or an area of jurisdiction. Thus we have a weak order $p > q$ to read "p is weaker than q". Then $>$ is to be transitive. Now define a relation on elements, a predicate $w(x,y)$. We say that $w(a,b)$ is true if for all p,q such that $p(a)$, $q(b)$ hold, $p > q$. In general, though, w will not define a transitive relation on elements. Now suppose that $p(b)$ is provable using only modus ponens,and that $p(a)$ is unprovable. (Notice that since T contains formal arithmetic for practical reasons, by Godel's theorem, unprovable statements will certainly occur. Then kal vahomer is

$$p(b) \ , \ w(a,b) \vdash p(a)$$

that is, we can assign truth values compatibly with some ordering. In Talmudic law, this rule is justified by refernce to Scripture again.

We now want to know whether kal vahomer is applicable to statements provable only by using it. We answer in the affirmative using a very ingenious argument as our justification.

Let a be the rule of kal vahomer, and b the rule

$$() \vdash q$$

when q is an axiom. This merely asserts that an axiom is also a theorem. Let $p(x)$ be "a statement provable by use of b can be used as a premiss in an inference by x.". Then $p(a)$ is true, and $p(b)$ is true. Now let $q(y)$ be "a statement provable by use of y can be used as a premiss in an inference by a.". Then $p(a) = q(b)$ and so is provable (indeed, an axiom, as far as we are concerned). Now $q(a)$ is certainly undecidable, and $p > p$ gives $w(a,b)$.

Thus by kal vahomer in this system, we derive $q(a)$, that is, the extended use of kal vahomer.

Please note that this is by no means a formal proof. It is merely arguing that if kal vahomer applies in the formal language, it is reasonable to admit it in the informal metalanguage: but this then validates the extended form in T.

We have thus achieved a reasonable extension of the predicate caculus. What is so remarkable is that this was originally argued in a system quite distinct from modern logic, in which, some 500 years ago, the importance of a metalanguage, and undecidable (or theku) statements was realised. Perhaps our modern mathematics is not quite so modern after all!

References.

A fuller statement and bibliography is in

P. Longworth   Confrontations with Judaism (1966),pp.171-196

There is a detailed account in

H. Guggenheimer "Uber ein bemerkenswertes logisches System in der Antike",Methods(1951)

ANSWERS(CONT.)

11) 769369397. D,M,B,C are arbitrary, except that B = C, and the rest are

A	E	H	I	L	N	O	P	R	S	T	V	

AEHILNOPRSTV
966969386376

12) The second argument is faulty. The events '2 | n', ... are independent in pairs, but not in threes, for if n is divisible by the two largest primes less than its square root, it will never be divisible by the third largest such prime.

13) Yes, after 334 units.

14) (a) 3. They are the number of letters in 'one','two',...
(b) 620 448 401 733 239 439 360 000 = 24! = 4!!
(c) 793. They are $3^n + (-2)^n$.
(d) 6. They are the decimal digits of $10 - \pi$.

Answers

by Colin Vout and Martin Brown

1) (a) A knot. (b) No. (c) Yes. (d) No.

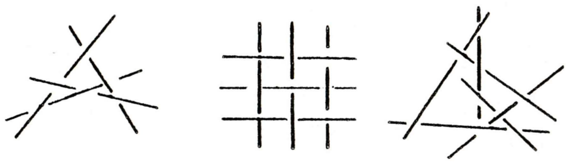
2) (i) $d(x) = (a_1 + 1)(\ldots)(a_n + 1)$

(ii) $S(x) = (1 - (-1)^{d(x)})/2$

(iii) $N(x) = \sum_0^x S(y)S(x-y) = [\sqrt{x}] - (x-3)/4 +$

$$\sum_0^{x}(-1^{d(x)-d(x-y)})/4$$

3) There are one frame with four and two with six sticks.



5) He takes longer going to work. He should aim for the point x minutes from the bridge, where x minmizes

$$\sqrt{(u^2 + v^2)} + (b + T - x)^2/4T$$

6) See page 24.

7) $X = 41$, $Y = 12$, $U = 49$, $V = 31$

8) The groups must have just one element of order 2, so the only possible groups are $C_{n-1} \times A$ where n and $|A|$ are odd and A is abelian .

9) Our hero does escape. He should head directly for one bad guy. When the angle subtended at him by this and every other bad guy is at least 2arcsin 3/4, which is bound to happen, he belts off along an angle bisector and lives happily ever after, composing problems for Eureka.

10) N = 8. The solution is

1 X_1^3	2 X_1^0
3 X_2^2	X_1^2