Eureka Editor
archim-eureka@srcf.net
The Archimedeans
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WA
United Kingdom

Published by The Archimedeans, the mathematics student society of the University of Cambridge

Thanks to the Betty & Gordon Moore Library, Cambridge

Eureka 55

Eureka

Eureka is the journal of the Archimedeans, which is the Cambridge University Mathematical Society. It is published approximately annually, but since it, like the Society, is run entirely by student volunteers, it is impossible to guarantee precise publication dates.

Subscriptions

A subscription account may be opened by sending at least ten pounds to the Business Manager at the address below; issues will be sent as they are published and you will be informed when your account is running low. Issues are charged at £2.50 plus the cost of postage. Back numbers of some of the more recent issues are available.

Electronic mail

The Society can receive electronic mail via the Internet, and can read most electronic media. Our electronic mail address is given below. There may be long delays in dealing with messages of any kind in the University vacations (especially June to October), and you are asked not to send large amounts of text electronically without prior warning.

Addresses

Correspondence, clearly marked "Editor" or "Business Manager", should be sent to:

Archimedeans—*Eureka*, Centre for Mathematical Sciences,
Wilberforce Road, Cambridge, CB3 0WA, England

Our electronic mail address is:

`soc-archim@lists.cam.ac.uk`

Committee of the Archimedeans, 2001—2002

EUREKA

Editor: Alan Bain — Number 55, April 2002

Table of Contents

ISSN 0071-2248

Editorial

Here is *Eureka* 55, albeit considerably later than was originally intended. At this point in time it seems worth pausing to consider the future of Eureka in a world considerably changed from that of 1939 when the first issue was produced. As an example, the world wide web provides for rapid and easy distribution of articles and ideas, with considerably less effort than going through a stream of people to eventually get a published article on paper. Yet there still seems a place for a printed journal of recreational mathematics, where originality is not required, and where interest and amusement for one's fellow undergraduates is the goal. I use the term recreational reservedly, neither meaning to suggest mathematics of a second rate nature, nor a certain kind of games-and-puzzles mathematics, which despite having a considerable following (amongst the Archimedeans and elsewhere) is not everyones' cup of tea. Indeed this issue aims to contain a variety of articles catering for a variety of tastes! The journal's future direction is ultimately in the hands of the current readership, and may well diverge from that taken in the past. If you have enjoyed reading this issue, is there some, possibly small, way in which you could contribute to the next one? What sort of articles do you like to read in Eureka—why not let the committee know!

I hope this issue is read more carefully than previous issues were read by the contributor who sent in an article entitled "Monomial distribution functions and their entropy", beginning "We report on a second order Fuchs type differential equation which admits a distribution function $f_\alpha(x)$ ...".

Cambridge, June 2001

This second printing incorporates a number of corrections; I am very grateful to those readers who drew the original errors to my attention.

Cambridge, April 2002

Acknowledgements

I should like to thank Dave Harris and Jon Peatfield for arranging computing and printing facilities in DAMTP which have been indispensable for the completion of *Eureka*. Mark Wainwright deserves thanks for freely offering criticism of the typography and layout of various drafts. Robert Beattie proofread many articles at very short notice and found numerous errors which would otherwise have passed undetected. Thanks are due to Sebastian Bleasdale for much encouragement and for the design and drawing of the cover. Any remaining typographical mistakes are of course mine.

The Committee 1996-2001

Several years have passed since *Eureka 54* in 1996, so it was felt by the current Archimedeans committee that in this issue all the committees who have held power in the intervening years should be recorded. As those involved will know, there have been also innumerable agents, who are not listed due to the lack of space. If you are interested in helping to run the Archimdeans, please get in touch with the current committee, whose details are provided inside the front cover of this issue.

Committee of the Archimedeans, 1996-1997

President	Mark Holland	Trinity
Vice-President	Rawnak-ul Islam	Trinity
Secretary	Adam Sutcliffe	Trinity
Junior Treasurer	Joanna Ashbourn	DAMTP
Registrar	Peter Benie	Downing
Publicity and Entertainments Manager	Rawnak-ul Islam	Trinity
Chronicler	Mark Holland	Trinity
Business Manager	Alan Bain	Trinity
Senior Treasurer	Dr W. T. Gowers	Trinity

Committee of the Archimedeans, 1997-1998

President	Alan Bain	Trinity
Vice-President	Paul Russell	Peterhouse
Secretary	John Haslegrave	Trinity
Junior Treasurer	Joanna Ashbourn	DAMTP
Registrar	Michelle Allen	Trinity
Publicity and Entertainments Manager	Ben Cohen	Trinity
Chronicler	Anne Harrison	New Hall
Business Manager	Alan Bain	Trinity
Senior Treasurer	Prof W. T. Gowers	Trinity

Committee of the Archimedeans, 1998-1999

President	Paul Russell	Peterhouse
Vice-President	Mark Boyd	Trinity
Secretary	John Haslegrave	Trinity
Junior Treasurer	Ben Cohen	Trinity
Registrar	Anne Harrison	New Hall
Publicity and Entertainments Manager	Robert Jones	Peterhouse
Chronicler	Dan Neale	Peterhouse
Business Manager	Ben Cohen	Trinity
Senior Treasurer	Prof W. T. Gowers	Trinity

Committee of the Archimedeans, 1999-2000

President	Shabnam Kadir	Trinity
Vice-President	Mark Wildon	Trinity
Secretary	Paul Huggins	Peterhouse
Junior Treasurer	Paul Russell	Peterhouse
Registrar	Robert Beattie	Trinity
Publicity and Entertainments Manager	Vacant	
Chronicler	Dan Neale	Peterhouse
Business Manager	Andrew McGregor	Caius
Senior Treasurer	Prof W. T. Gowers	Trinity

Committee of the Archimedeans, 2000-2001

President	Andrew McGregor	Caius
Vice-President	Robert Beattie	Trinity
Secretary	Jason Holt	Caius
Junior Treasurer	Jochen Mundinger	Caius
Registrar	Lucy Devlin	Caius
Publicity and Entertainments Manager	Dan Wood	Johns
Chronicler	Florian Herzig	Caius
Business Manager	Sebastian Bleasdale	Clare
Senior Treasurer	Dr I. B. Leader	Trinity

Tatami Tilings

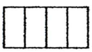
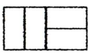
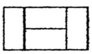
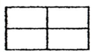
Alex Barnard

1. Introduction

In the previous issue of Eureka [1], Adam Chalcraft posed a question about the possibility of tiling an n by m rectangle with 2 by 1 rectangles under a restriction on the placement of the 2 by 1 rectangles. For the benefit of those who did not see the original question it is reproduced here.

Consider an n by m rectangle with n,m both positive integers and nm even. Can this rectangle be tiled with 2 by 1 rectangles such that no four rectangles share a common corner? If not, what is the smallest rectangle which can not be so tiled?

Thus the first three of the following configurations is allowed, whereas the last is not:

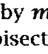
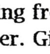
(a)  (b)  (c)  (d) 

This article aims to completely answer this question. The final results of which a proof will be given later are (a) that for each height n there is a certain value of m after which all possibilities can be tiled, and (b) for every $n > 6$ there is a rectangle which can not be tiled, and (c) necessary and sufficient conditions for a rectangle to be tiled will be given.

Throught I shall assume that the rectangle is of size n by m where n,m are both positive integers such that $1 < n \leqslant m$ and nm is even. The case $n = 1$ is trivially possible for any even m.

2. Definitions

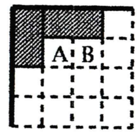
Following the title of the original article I shall call a rectangle which can be tiled a *Tatami Rectangle.* An arrangement of the 2 by 1 rectangles in one of the following forms will be called a *seed* (the first is a *horizontal seed,* the second a *vertical seed*):

(a)  (b) 

A *diagonal* of the n by m rectangle is a line coming from one of the corners of the rectangle which bisects the angle at that corner. Given a diagonal of the rectangle, a seed is *barely-on-diagonal* if the centre of exactly one of the four 1 by 1 squares making up the seed is on the diagonal. Otherwise the seed is *off-diagonal.*

LEMMA 1. *In any tiling of the rectangle there is a seed which is on-diagonal.*

Proof. Assume that the lemma is false. By assumption there is not a seed in the top left hand corner, and so by reflection we may assume the tiling to be like the following diagram:



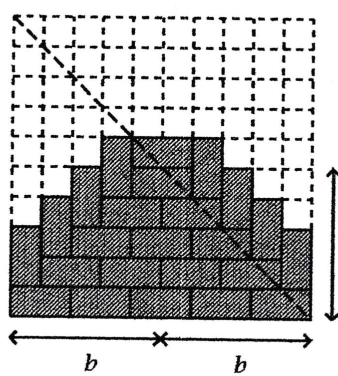
Now consider the square A. If a 2 by 1 rectangle is placed horizontally on this square then we will have formed a seed which is on-diagonal, so we must have the 2 by 1 rectangle which covers square A being vertical. Similarly we must have the rectangle covering B horizontal. Hence we are now in an equivalent position to before and we can repeat the above argument ad infinitum. This is a contradiction as n and m are finite and so the lemma is true. □

It is then easy to show by careful consideration of this lemma:

LEMMA 2. *If the rectangle is Tatami and the nearest seed is barely on diagonal for the diagonal from the top-left corner then it is horizontal if it is above the diagonal and vertical if it is below the diagonal.*

Now the reason why I called the 2 by 2 block a *seed* will become clear. If a seed is placed somewhere on the n by m rectangle there remains very little choice as to where to put other rectangles if the *no four sharing a corner* rule is used. So once the seed has been 'planted' then it will cause a tiling to 'grow' from it. We shall now look at what happens when we place the on diagonal seed for the top-left corner with its centre a distance b from the bottom edge. Assume that the rectangle is orientated so that n is its height.

Consider a horizontal seed which is strictly on-diagonal. Obviously the tiling would have to look like this:

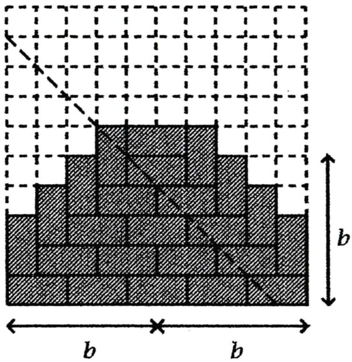


So to fit into the rectangle the amount that the pattern extends to the left must not be more than is available in the rectangle. So $b \leqslant n - b$. Hence $2b \leqslant n$.

Similarly extending the pattern to the top-left corner we see that $n - b - 1 \leqslant b$. Hence $2b > n - 1$.

So combining these two results for b we see that $n - 1 \leqslant 2b \leqslant n$.

If the seed is horizontal but barely on diagonal then the reader can check by following to the top-left corner, that the tiling would look like:



As before we get $b \leqslant n - b + 1$. Hence $2b \leqslant n + 1$. Also we get $n - b \leqslant b$. Hence $2b \geqslant n$. And so combining these two results for b we see that $n \leqslant 2b \leqslant n + 1$.

Similarly we get results for a vertical seed, and all these results are sumarised below:

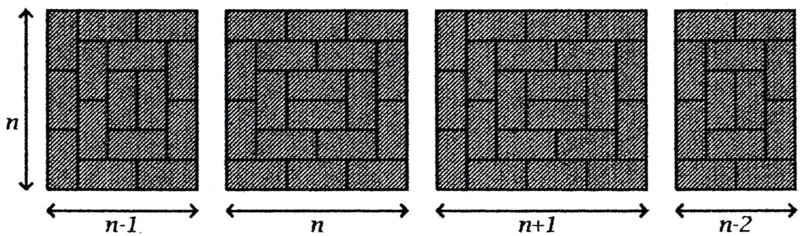
PROPOSITION 1. *If the rectangle is Tatami and the on-diagonal seed from the top-left corner has its centre a distance b from the bottom edge then b must satisfy the following restrictions:*

(i) If the seed is horizontal strictly on diagonal or vertical barely on diagonal then $n - 1 \leqslant 2b \leqslant n$,

(ii) If the seed is strictly on-diagonal or horizontally barely on diagonal then $n \leqslant 2b \leqslant n + 1$.

Now if n is odd then these conditions mean that the barely on-diagonal seeds are in fact strictly on-diagonal for the diagonal from the bottom left hand corner. So in the case of n odd we may assume that the seed is strictly on diagonal. If n is even then these conditions mean that a seed must lie exactly half way between the top and bottom edges of the rectangle.

So if n is even then the four possibilities look like:



So after growing the seed we are left with a smaller n' by m rectangle which is to be tiled. Now we can apply the above arguments to this smaller rectangle, if

$m > n'$. So we see that the tiling will be made up of the four patterns above with perhaps a small rectangle untiled at the right hand end of the n by m rectangle. In fact looking more closely at the above patterns it is easy to see that any of the four may come at the left hand end, then there will be a series of copies of the first and third case (as they are the only ones which match up at the left hand side). Now consider the gap remaining at the right hand side, by the above tiling procedure it can have a width of at most $n - 2$. If it is simply one square wide then it can easily be tiled up, however if it is any wider then a pattern of width $n-1$ will grow. Hence, if the rectangle is assumed to be Tatami this second case is not possible. Using this information we now show:

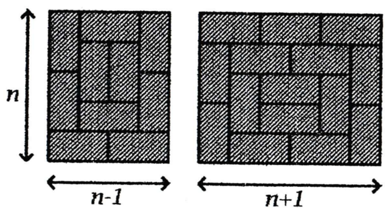
THEOREM 1. *An n by m rectangle with n even and $m \geqslant n$ is Tatami if and only if there is a positive integer k such that $k(n-1) - 1 \leqslant m \leqslant k(n+1) + 1$.*

Proof. Consider what happens if we start with any of the four possibilities and follow this by k repetitions of the first shape and possibly a column of length 1 at the end. This gives a tiling of an n by m rectangle with m in the range $(k+1)(n-1) - 1$ to $(k+1)(n-1) + 3$. Now if at any point we use a copy of the third shape instead of the first we simply incrase the width by two. Hence by using k repetitions of either the first or third shapes we can get m in the range $(k+1)(n-1) - 1$ to $(k+1)(n-1) + 3 + 2k$. Or simplifying the expressions we get the range $K(n-1) - 1$ to $K(n+1) + 1$ with K a positive integer. $\square$

So for example we can tile all rectangles with one side of length 6, whereas we can not tile an 8 by 11 rectangle. Also note, from the way the upper and lower constraints are increasing in the inequality, there will be a value of k for which the lower constraint is less than the previous upper constraint and the same will be true for any larger k. Thus eventually all integers can be represented in the way required by Theorem 1. The value of k is given by $k(n-1) - 1 \leqslant (k-1)(n+1) + 2$ which gives $k \geqslant (n-2)/2$. So we can show:

COROLLARY 1. *For an n by m rectangle, with $6 < n \leqslant m$ and n even, the largest m for which the rectangle is non-tatami is given by $m = (n^2 - 5n)/2$.*

Now consider what happens if n is odd. The two (not four as we may assume that the seed is strictly on-diagonal) possibilities for the left hand side are:



So, by an argument similar to the one used for n even, we see that the tiling is made up of a sequence of these shapes placed side by side (possibly after having been turned upside down allowing it to match at the left hand side). As before there is possibly a small untiled rectangle at the right hand side, and its width is at most $n-2$. If there is any gap a tiling of width $n-1$ will grow. So if we assume the rectangle to be Tatami there can not be any gap at the right hand side. So we can now show:

THEOREM 2. *An n by m rectangle, with n odd, m even and $n \leqslant m$ is Tatami if and only if there is a positive integer k such that $k(n-1) \leqslant m \leqslant k(n+1)$.*

Proof. *As before.* □

COROLLARY 2. *For an n by m rectangle, with $6 < n \leqslant m$, n odd and m even, the largest m for which the rectangle is non-Tatami is given by $m = (n^2 - 4n - 1)/2$.*

Note that the two theorems can easily be combined to give:

THEOREM 3. *An n by m rectangle with $n \leqslant m$ is Tatami if and only if there is a positive integer k such that*

$$\frac{m-1}{n+1} \leqslant k \leqslant \frac{m+1}{n-1}.$$

So from the above results it is easy to see that the smallest rectangle which can not be tiled in the required way is a 7 by 10 rectangle.

3. References

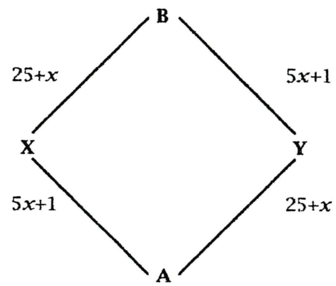
[1] Chalcraft, A. (1996), *Tatami Mats*, Eureka 54.

A Small Road Network

Mark Wainwright

1. The set-up

Let's consider a network of roads between two sites, **A** and **B**. Each road has a cost associated with travelling on it, and this cost is some function of the number of cars on the road: more cars mean longer journeys and more accidents, for example. The function will in general be complex, but we shall keep things simple by having a simple linear function for each road. The network looks like this:



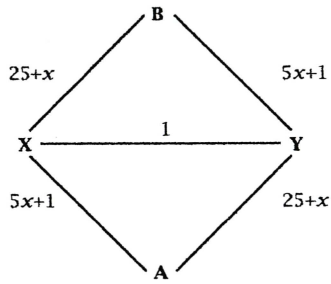
The four roads are marked with their cost functions, x being the number of cars on the road. **A** is a residential area, and **B** an industrial area, so the inhabitants of **A** commute to **B** every day. **A** happens to be quite small: it has a population of six. So each day, six cars set off at half-past eight from **A** to get to **B**.

Let's imagine each car in turn setting off from **A**. Each driver can see how much traffic is already on each road, and choose their own route accordingly. The first car could go either way; let's say it goes by the route to the right. Then the second car will go to the left. Successive pairs of cars will split one each way, and there will be three cars using each road. So the cost for each car will be $26 + 6 \times 3 = 44$.

In real life, on the one hand drivers may not have perfect information about the traffic already on the road; and on the other hand they may take into account how much traffic they expect to be on the road later (from past experience, say). Neither of these facts significantly changes the analysis, as we'll see later.

2. The politician

The constituency containing **A** is marginal, and the sitting MP needs to make sure of those extra six votes, so she applies pressure to the planning authorities to build an extra road to help the commuters. It is a good fast road across the middle of the network between points **X** and **Y**.



Where will the first car go now? Clearly it will take the route **A–X–Y–B**: this has a cost of 13, until someone else starts using the road. Hot on the trail comes Car 2. **A–X–Y–B** is still best (cost 23). Car 3 starts out and also takes **A–X–Y–B** (cost 33). When Car 4 sets out, **A–X–B**, would cost 47 (there are three cars already using **A** → **X**), so he too takes **A–X–Y–B**, whose cost is only 43.

By the time Car 5 sets out, **A–X–B** (cost: 52) is cheapest, so he takes that route, and Car 6 takes **A–Y–B** with the same cost. The final cost for each of cars 1–4 is $(2 \times 5 \times 5) + 3 = 53$, so everyone's journey cost comes out to at least 52.

How has life improved by the addition of the new road? Well, before it existed, the cost of getting to work was only 44, so everyone's travelling cost has gone up by 18 per cent. But they all have more choice about which route to take to work, so they are all much happier.

3. The extent of the problem

We made an important assumption above when modelling the way the commuters decide which route to take. We used the *greedy algorithm*: put the cars on the road one at a time, and let each take the route that currently has lowest cost. This assumption may have been wrong, so this is a good point to look at the network a little more closely.

- In our original network, the solution we found (three cars going each way) was *stable*: once the cars are arranged in this way, no-one wants to change route. If any car unilaterally changes to a different route, that car's journey cost will increase.
- Furthermore, it was the only stable solution. If we arrange the cars a different way—four to the left and two to the right, say—one driver will soon notice that he could have lowered his own journey cost by taking a different route.
- So, if we start with any arrangement of cars on routes, but let the cars change to a different route if it is cheaper, they will soon arrive at the stable solution, which happens to be the one we found by the greedy algorithm.

What is particularly interesting is that these three properties also apply to the modified network. The solution given by the greedy algorithm, with four cars taking **A–X–Y–B**, and one each **A–X–B** and **A–Y–B**, is stable, and it is the only stable solution. If the six cars start off by using their old route to work—three each to the left and the right—one driver will soon notice that he can improve his time by cutting across between **X** and **Y**, and gradually others will also change until the cars are arranged in the new formation where everyone's journey is more expensive.

You are strongly encouraged to convince yourself of these facts by experimenting with different arrangements. If you're feeling adventurous, you might like to try making up your own networks.

4. The analysis

How can it happen that everyone is careful to take the cheapest available route, and yet ends up worse off? How can adding a road to a network slow everyone down? The network is very small, so we should be able to get a handle on what is going on.

The key is how the cost of using a road varies with the number of cars. Using our linear functions, we can imagine the constant term as representing the road's length, and the term in x an indication of how prone the road is to congestion. (This idea supposes that the 'cost' of a journey is purely a matter of how long it takes.) Then the politician's new road allowed a short-cut between the two north-westerly roads, which are shorter than the alternatives, but suffer from substantially worse congestion.

The result is that each driver in turn takes this slightly quicker route, at a small saving for himself, but causing substantial extra delay to a number of other drivers. By the time everyone has decided whether to take the short cut, the total congestion gained by the system far outweighs the savings in distance.

The modified network is really a classic example of the Prisoner's Dilemma (**PD**), a much-mooted paradox with many applications, but in general no solution. Adding the extra road gave the commuters the opportunity to 'defect', in **PD** terms.

5. The reality

This property of networks is called Braess's paradox, and it is not just a theoretical construct. Our network was very, very simple; it had only one journey, only five roads and the cost functions were as simple as could be—linear functions. Real networks such as road networks or telecommunications networks are far, far more complicated. This makes them more likely, not less, to exhibit paradoxes like this. They are also harder to analyse.

Imagine trying to arrange routing for a telecommunications network (routing telephone calls, data links, and so on). 'Take the quickest-looking route' seems like a good idea, but how can we prevent bad congestion due to Braess's paradox? One possibility is for a central router to know about all the connections being made and calculate a global optimum. Unfortunately, this isn't likely to work very well. For one thing, the calculation itself would be impracticably hard. More seriously, connections to and from the central router would themselves become heavily congested, and also the system would be prone to point failure—if the router breaks, everything breaks. More practical solutions involve finding rules to apply locally, not globally. The result may not be the best possible but it will be fairly good and, importantly, robust. What forms such rules should take is a difficult problem and the subject of active current research.

6. The moral

Our road network was a very, very simple model of a free market. The consumers (drivers) were given a free choice of which route to take, in the happy supposition that this would in time lead them to settle into a pattern of the most efficient journeys possible.

Like any truly unregulated free market, it was a miserable failure. The problem was not merely that consumers cannot be relied on to take the most advantageous course, or even that the few benefited at the expense of the many, or vice versa. Even when all the consumers did their civic best to grab what was in it for themselves, every single one ended up worse off than they started. What is more, our system could hardly have been simpler; real markets are very much more complex, and so more prone to this kind of failure. Regulation in a market exactly corresponds to the kind of locally-applied rules that we imagined above.

Adam Smith, the famous and influential 18th-century economist, believed that an unregulated market where everyone made well-informed decisions would lead, by a mathematical necessity, to the most equitable and satisfactory situation. He called this principle the 'invisible hand' of the market. Unfortunately, Smith wasn't a mathematician, so he never tried to prove this 'obvious' but false result. If he had done, perhaps he would have found Braess's paradox over two centuries ago, and discovered the power of market forces to turn a pleasant morning spin into a total snarl-up.

References

Braess's paradox was first described in 1968 (see [2]). Frank Kelly's paper [1] on Network routing provides background on some of the areas where Braess's paradox is a problem, and indicates some of the ways one might go about trying to solve it. See his bibliography for further reading; for more on the Prisoner's Dilemma, see the excellent article [3].

[1] Kelly, F. (1991). *Network Routing*, Phil. Trans. Royal Soc., **A337**, 343–367
[2] Braess, D. (1968). *Über ein Paradoxon aus der Verkehrsplanung*, Unternehmensforschung **12**, 258–268.
[3] Murphy, K. (1992). *The Prisoner's Dilemma*, Eureka **51**.

Qarch Problems

As some years have elapsed since the last issue of Qarch was published in 1992, the statements of the problems to which solutions are provided later in this issue of Eureka are included here. Problems for which solutions are included are indicated by an asterisk beside the problem number.

49*. Tiling Problem.

For which $n \in \mathbb{N}$ does there exist a cuboid with positive volume which can be tiled the the shape:

,

which in n-dimensions should be regarded as five n-cubes in the above arrangement.

53*. Generalised Borromean Rings Problem.

It is well known that three rings can be arranged in such a way that although they cannot be pulled apart, if any one of them is cut then they can all be pulled apart. The ${}^{n}B_{r}$ problem (for $0 \leqslant r < n$) is to find an arrangement of n rings in $\mathbb{R}^3$ such that if any r are cut then they all fall apart, but that if any $r - 1$ are cut then the remainder do not fall apart, or prove that no such arangement exists.

56*. Fair Dice.

A fair die is an n-dimensional convex polytope all of whose faces ($n - 1$ dimensional) are equivalent in these sense that the group of symmetries acts transitively on faces. For which m and n do fair n-dimensional m-sided dice exist? A resonable place to start would seem to be $n = 3$, followed by $n = 4$, and 5 in the special case where the faces are simplices.

64. Colouring Trees.

Given k spanning trees on $1, \dots, n$ and a k-colouring of one of them, show that the remaining $k - 1$ trees can be k-coloured such that for each colour the set of edges of that colour forms a spanning tree.

65. Unlucky Lotteries.

How many tickets are required to guarantee that a prize is won in the National Lottery? (On each ticket you select six numbers from $1, \dots, 49$. Six numbers are selected at random and you win a prize if you have at least three correct.) More generally if the numbers are chosen from $1, \dots, n$ how does the number $f(n)$ of tickets required vary?

66. A Room and a Half.

How many non-overlapping unit square tiles can be placed in a square room with sides 1000.5 units? More generally what is the maximum number of tiles that can be placed in a room with sides $n + \frac{1}{2}$? Alternatively, letting

$$\epsilon_n = \inf\{\epsilon : \text{more than } n^2 \text{ tiles can be placed in a } (n + \epsilon) \times (n + \epsilon) \text{ room}\},$$

how does ϵ_n vary?

67. Spanning Trees.

For a graph G on $1,\ldots,n$ let $m(G)$ denote the number of spanning trees of $1,\ldots,n$ that are subgraphs of G. Given any two such graphs F and G prove that

$$m(F)m(G) \geqslant m(F \cap G)m(F \cup G).$$

68. Hilbert Space Filling Curves.

Given a Hilbert space H does there exist a continuous surjective map $\mathbb{R} \to H$? What if H is separable and endowed with the weak topology? [The weak topology is the weakest topology making all the functions $x \mapsto <x, y>$ continuous.]

69. Blockbusters?

Let K be a triangulation of the hypercube I^{g+r}. Divide ∂K into two parts K_g and K_r with $K_g = \partial I^g \times I^r$ and $K_r = \partial I^r \times I^g$ (where ∂ denotes boundary). Two players, Red and Green, take turns to colour the vertices o K their colour. When they have finished Red wins if there is an r-surface, coloured red, spanning K_r (i.e., there exists a non-zero element of $H_r(K, K_r)$ made up of simplices with all vertices red). Prove that exactly one player wins.

70. The 2^n Bricks.

Define a brick to be a set of the form $B_1 \times B_2 \times \ldots \times B_n$ and a sub-brick to be a subset of a brick of the form $b_1 \times b_2 \times \ldots \times b_n$ with $b_i < B_i$ for $1 \leqslant i \leqslant n$. Show that any partition of a brick into sub-bricks requires at least 2^n sub-bricks. (It is clear that this bound can be attained.)

71. Decomposing Matrices.

Given a matrix $A \in \mathrm{SO}(n)$, (i.e., $A^T A = I$, $\det A = 1$) show that, when A is written as

$$A = \begin{pmatrix} A_1 & B \\ C & A_2 \end{pmatrix}$$

where A_1 is an $r \times r$ matrix and A_2 is an $(n-r) \times (n-r)$ matrix, $\det(A_1) = \det(A_2)$.

72. Homeomorphisms

Find an elementary proof that $\mathbb{R}^7$ and $\mathbb{R}^7 - \{0\}$ are not homeomorphic.

73. Knotting Cubes.

Suppose a cube is on an infinite plane and is free to roll over any edge. Mark one of its corners and trace its path as the cube rolls around the plane. Suppose now that the corner comes back to its original position without the path intersecting itself; then the path could be knotted. What knots can be obtained in this way? Now suppose that two corners are marked. What links can be obtained? [A link is two loops knotted in some way.] If three corners are marked can Borromean rings be traced out?

74. Free Groups.

Let P be the group with presentation $\langle x, y : x^p = y^q \rangle$ where p and q are coprime. Show that the commutator subgroup is free on an even number of generators.

divisor of m_i ($i = 1,2$); conversely, if d_i is a divisor of m_i ($i = 1,2$) then $d_1 d_2$ is a divisor of $m_1 m_2$. Hence

$$\begin{aligned} D(m_1 m_2) &= \sum_{d \mid m_1 m_2} d \\ &= \sum_{\substack{d_1 \mid m_1 \\ d_2 \mid m_2}} d_1 d_2 \\ &= \Bigl(\sum_{d_1 \mid m_1} d_1\Bigr)\Bigl(\sum_{d_2 \mid m_2} d_2\Bigr) \\ &= D(m_1) D(m_2), \end{aligned}$$

as required.

1.2. Classification of even perfect numbers

It is easy to classify the even perfect numbers: they are precisely those numbers $2^{r-1}(2^r - 1)$ where $r \geqslant 2$ and $2^r - 1$ is prime. (Of course, computing which values of r make $2^r - 1$ prime is itself a hard problem.) The first three perfect numbers are $2 \times 3 = 6$, $4 \times 7 = 28$, and $16 \times 31 = 496$.

In one direction, suppose that $r \geqslant 2$ and $2^r - 1$ is prime: then by 1.1,

$$\begin{aligned} D(2^{r-1}(2^r - 1)) &= D(2^{r-1}) D(2^r - 1) \\ &= \left(1 + 2 + 2^2 + \cdots + 2^{r-1}\right)(1 + 2^r - 1) \\ &= (2^r - 1)\, 2^r \\ &= 2\left[2^{r-1}(2^r - 1)\right], \end{aligned}$$

so $2^{r-1}(2^r - 1)$ is an even perfect number.

In the other direction, suppose that n is an even perfect number. Write $n = 2^s m$ where $s \geqslant 1$ and m is odd: then n being perfect says that

$$D(2^s m) = 2 \times 2^s m,$$

i.e.

$$\left(2^{s+1} - 1\right) D(m) = 2^{s+1} m,$$

i.e.

$$\left(2^{s+1} - 1\right)(D(m) - m) = m. \qquad (*)$$

Hence $D(m) - m$ is a divisor of m, and since

$$2^{s+1} - 1 > 2^{0+1} - 1 = 1,$$

it is a proper divisor of m. But $D(m) - m$ is by definition the sum of the proper divisors of m, so $D(m) - m$ is the unique proper divisor of m. Thus m is prime and $D(m) - m = 1$, and by $(*)$, the latter means that $m = 2^{s-1} - 1$. So $n = 2^s(2^{s+1} - 1)$ with $s \geqslant 1$ and $2^{s+1} - 1$ prime, as required.

2. Definition and First Examples of Perfect Groups

In this section we define the notion of a perfect group, and search for examples among some of the well-known families of groups (symmetric, alternating, ...). In fact, the only examples of perfect groups we will find are cyclic, although by section 3 we will have developed enough theory to be able to exhibit some more interesting examples.

Of the examples below, only the cyclic groups (2.1) and the symmetric and alternating groups (2.2) will be needed later on.

The reader is reminded that a *normal subgroup* of a group G is a subset of G which is the kernel of some homomorphism from G to some other group; equivalently, it is a subgroup N of G such that $gng^{-1} \in N$ for all $n \in N$ and $g \in G$. We write $N \trianglelefteq G$ to mean that N is a normal subgroup of G. From here on, 'group' will mean 'finite group'.

If G is a group, define $D(G) = \sum_{N \trianglelefteq G} |N|$, the sum of the orders of the normal subgroups of G, and say that G is *perfect* if $D(G) = 2|G|$.

2.1. Example: cyclic groups

Let C_n be the cyclic group of order n. Then C_n has one normal subgroup of order d for each divisor d of n, and no others, so $D(C_n) = D(n)$ and C_n is perfect just when n is perfect. Thus perfect groups provide a generalization of the concept of perfect numbers, and C_6, C_{28} and C_{496} are all perfect groups.

2.2. Example: symmetric and alternating groups

None of the symmetric groups S_n or alternating groups A_n is perfect. If $n \geqslant 5$ then A_n is simple and the only normal subgroups of S_n are 1, A_n and S_n, so $D(A_n)$ and $D(S_n)$ are too small. For $n \leqslant 4$, we have

$$\begin{aligned} D(A_1) &= 1, & D(S_1) &= 1, \\ D(A_2) &= 1, & D(S_2) &= 1+2=3, \\ D(A_3) &= 1+3=4, & D(S_3) &= 1+3+6=10, \\ D(A_4) &= 1+4+12=17, & D(S_4) &= 1+4+12+24=41. \end{aligned}$$

2.3. Example: p-groups

A (finite) *p-group* is a group of order p^r, where p is prime and $r \geqslant 0$. Lagrange's Theorem says that the order of any subgroup of a group divides the order of the group, so if G is a p-group then $D(G) \equiv 1 \pmod p$. Hence no p-group is perfect.

2.4. Example: dihedral groups

Let E_{2n} be the dihedral group of order $2n$: that is, the group of all isometries of a regular n-sided polygon. Of the $2n$ isometries, n are rotations (forming a cyclic subgroup of order n) and n are reflections. We examine the cases of n odd and n even separately.

In the case when n is odd, all reflections are in an axis passing through a vertex and the midpoint of the opposite side, and any reflection is conjugate to any other by a suitable rotation. Thus if $N \trianglelefteq E_{2n}$ and N contains a reflection, then N contains all reflections; but $1 \in N$ too, so $|N| \geqslant n+1$, so $N = E_{2n}$. So any proper normal subgroup is inside the rotation group C_n; conversely, any (normal) subgroup of C_n is normal in E_{2n}. Thus

$$D(E_{2n}) = D(C_n) + 2n,$$

and E_{2n} is perfect if and only if n is a perfect number.

In the case when n is even, the reflections split into two conjugacy classes, R_1 and R_2, each of size $n/2$: those in an axis through two opposite vertices, and those in an axis through the midpoints of two opposite sides. Write $C_{n/2}$ for the group of rotations by 2 or 4 or ... or n vertices, a subgroup of E_{2n} which is cyclic of order $n/2$. Then we can show that the smallest subgroup of E_{2n} containing R_i is $R_i \cup C_{n/2}$, for $i = 1$ and 2. Moreover, $R_i \cup C_{n/2}$ is of order n, i.e. index 2, therefore normal in E_{2n}. So we have two different normal subgroups, $R_1 \cup C_{n/2}$ and $R_2 \cup C_{n/2}$, of order n. We also have the normal subgroups $\{1\}$ and E_{2n}, hence

$$D(E_{2n}) \geqslant 1 + n + n + 2n > 4n$$

and E_{2n} is not perfect.

In summary, the perfect dihedral groups are in one-to-one correspondence with the odd perfect numbers—so it is an open question as to whether there are any.

3. Multiplicativity

We proved in 1.1 that the function $D(n)$, on numbers n, was multiplicative. The aim of this section is to prove an analogous result for groups, and then to give some examples of nonabelian perfect groups by using this result.

Some difficulties are present for the reader not acquainted with composition series and the Jordan-Hölder Theorem. However, it is still possible for him or her to understand an example (3.3) of a nonabelian perfect group, provided that the following fact is taken on trust: if G_1 and G_2 are groups whose orders are coprime, and $G_1 \times G_2$ their direct product, then $D(G_1 \times G_2) = D(G_1)D(G_2)$. This done, the reader may proceed to 3.3 straight away.

The Jordan-Hölder theorem states that any two composition series for a group G have the same set-with-multiplicities of factors, up to isomorphism of the factors. I shall write this set-with-multiplicities as $c(G)$, and use $+$ to denote the disjoint union (or 'union counting multiplicities') of two sets-with-multiplicities. Thus if

$$c(G) = \{C_2, C_2, C_5\} \text{ and } c(H) = \{C_2, A_6\}$$

then

$$c(G) + c(H) = \{C_2, C_2, C_2, C_5, A_6\}.$$

We will use the fundamental fact that if if $K \trianglelefteq X$ then $c(X) = c(X/K) + c(K)$.

A pair of groups will be called *coprime* if they have no composition factor in common; alternatively, we will say that one group is *prime to* the other. (In particular, if two groups have coprime orders then they are coprime.) We will prove that D is *multiplicative*: that is, if G_1 and G_2 are coprime then $D(G_1 \times G_2) = D(G_1)D(G_2)$. First of all we establish the group-theoretic analogue of a number-theoretic result from section 1—namely, the second sentence of 1.1.

PROPOSITION 3.1. *Let G_1 and G_2 be coprime groups. Then the normal subgroups of $G_1 \times G_2$ are exactly the subgroups of the form $N_1 \times N_2$, with $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$.*

Proof. If $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ then $N_1 \times N_2 \trianglelefteq G_1 \times G_2$; conversely, let $N \trianglelefteq G_1 \times G_2$. Write $\pi_i : G_1 \times G_2 \to G_i \quad (i = 1, 2)$ for the projections, and regard G_1 as a normal subgroup of $G_1 \times G_2$ by identifying it with $G_1 \times \{1\}$, and similarly G_2. We have

$$\pi_1 N \cong \frac{N}{\ker(\pi_1|_N)} = \frac{N}{G_2 \cap N},$$

so by the 'fundamental fact' above,

$$c(N) = c(\pi_1 N) + c(G_2 \cap N);$$

and therefore by symmetry

$$c(\pi_1 N) + c(G_2 \cap N) = c(\pi_2 N) + c(G_1 \cap N).$$

But $c(\pi_i N) \subseteq c(G_i)$ and G_1 and G_2 are coprime, so $c(\pi_1 N)$ and $c(\pi_2 N)$ have no element in common; similarly $c(G_i \cap N) \subseteq c(G_i)$, so $c(G_1 \cap N)$ and $c(G_2 \cap N)$ have no element in common. Hence $c(\pi_i N) = c(G_i \cap N)$. We also know that $c(X)$ determines the order of a group X and that $G_i \cap N \subseteq \pi_i N$, so in fact $G_i \cap N = \pi_i N$. Thus

$$\pi_1 N \times \pi_2 N = (G_1 \cap N) \times (G_2 \cap N) \subseteq N,$$

and as always

$$N \subseteq \pi_1 N \times \pi_2 N,$$

so $N = \pi_1 N \times \pi_2 N$, with $\pi_i N \trianglelefteq G_i$. □

COROLLARY 3.2. *D is multiplicative.*

Proof. This is a direct analogue of 1.1. For by proposition 3.1,

$$\begin{aligned} D(G_1 \times G_2) &= \sum_{\substack{N_1 \trianglelefteq G_1 \\ N_2 \trianglelefteq G_2}} |N_1 \times N_2| \\ &= \sum_{N_1 \trianglelefteq G_1} \sum_{N_2 \trianglelefteq G_2} |N_1||N_2| \\ &= D(G_1)D(G_2). \end{aligned}$$

□

We can now exhibit three nonabelian perfect groups.

3.3. Example: $S_3 \times C_5$

The group $S_3 \times C_5$, of order 30, is perfect. For S_3 and C_5 have coprime orders (6 and 5), so are coprime, so

$$\begin{aligned} D(S_3 \times C_5) &= D(S_3)D(C_5) \\ &= (1 + 3 + 6) \times (1 + 5) \\ &= 60 \\ &= 2\,|S_3 \times C_5|. \end{aligned}$$

3.4. Example: $A_5 \times C_{15128}$

We present this example (of order 907 680) along with the method by which it was found. Firstly, A_5 is a simple group of order $5!/2 = 60$. Now, let us try to find a perfect group G of the form $G = A_5 \times G_1$ where G_1 is some group prime to A_5. Since

$$D(A_5)/\,|A_5| = 61/60,$$

we need to find a G_1 such that

$$D(G_1)/\,|G_1| = 120/61.$$

Let us look for such a group G_1 amongst those of the form $G_1 = C_{61} \times G_2$, where G_2 is prime to C_{61} and A_5. Since

$$D(C_{61})/\,|C_{61}| = 62/61,$$

we need to find a G_2 such that

$$D(G_2)/\,|G_2| = 120/62 = 60/31.$$

In turn, let us look for such a group G_2 amongst those of the form $G_2 = C_{31} \times G_3$, where G_3 is prime to C_{31}, C_{61} and A_5. Since

$$D(C_{31})/\,|C_{31}| = 32/31,$$

we need to find a G_3 such that

$$D(G_3)/\,|G_3| = 60/32 = 15/8.$$

This is satisfied by $G_3 = C_8$, and the groups A_5, C_{61}, C_{31} and C_8 are pairwise coprime. Thus if

$$\begin{aligned} G &= A_5 \times C_{61} \times C_{31} \times C_8 \\ &= A_5 \times C_{61\times 31\times 8} \\ &= A_5 \times C_{15128} \end{aligned}$$

then G is perfect.

3.5. Example: $A_6 \times C_{366776}$

By the same technique we get this next example, of order 132 039 360. This time, we start with the simple group A_6 of order $6!/2 = 360$, and the sequence of groups A_6, C_{361}, C_{127}, C_8 'works' in the sense of the previous example. The details are left to the reader; note that $361 = 19^2$ and that 127 is prime.

4. The Abelian Quotient Theorem: Proof by Counting

In each of the next two sections we present a separate proof of our main classification result, the abelian quotient theorem. The two proofs have rather different flavours, and each produces its own insights, which is why both are included. We start with the more elementary of the two.

An *abelian quotient* of a group G is just a quotient of G which is abelian. That is, it's an abelian group A for which there exists a surjective homomorphism $G \to A$; alternatively, it's an abelian group isomorphic to G/K for some normal subgroup K of G. We will prove:

THEOREM 4.1 (ABELIAN QUOTIENT THEOREM). *If G is a group with $D(G) \leqslant 2|G|$ then any abelian quotient of G is cyclic.*

This result has the following corollaries, the second of which says that abelian perfect groups 'are' just perfect numbers:

COROLLARIES 4.2.
(a) If G is a perfect group then any abelian quotient of G is cyclic.
(b) The perfect abelian groups are precisely the cyclic groups C_n of order n with n perfect.

Proof. Part (a) is immediate. For (b), if A is perfect abelian then A is an abelian quotient of the perfect group A, hence A is cyclic. But we have already seen (2.1) that the perfect cyclic groups correspond exactly to the perfect numbers. □

(Those who know about such things will recognize that the theorem could be stated more compactly in this way: if G is a group with $D(G) \leqslant 2|G|$ then G^{ab} is cyclic. Here G^{ab} is the *abelianization* of G: it is an abelian quotient of G with the property that any abelian quotient of G is also a quotient of G^{ab}. In particular, if A is abelian then $A^{ab} \cong A$, which is how we would deduce corollary 4.2(b) from this formulation.)

The proof of the abelian quotient theorem given in this section uses two ingredients. The first is a new way of evaluating $D(G)$:

LEMMA 4.3. *For any group G,*

$$D(G) = \sum_{g \in G} |\{\text{normal subgroups of } G \text{ containing } g\}| .$$

Proof. We have

$$\begin{aligned} D(G) &= \sum_{N \trianglelefteq G} |N| \\ &= |\{(N, g) : N \trianglelefteq G, g \in N\}| \\ &= \sum_{g \in G} |\{N : N \trianglelefteq G, g \in N\}| . \end{aligned}$$

□

The second ingredient is the 'standard' fact that the inverse image (under a homomorphism) of a normal subgroup is a normal subgroup. For let $\pi : G_1 \to G_2$

be a homomorphism of groups, and let $N \trianglelefteq G_2$. Then N is the kernel of the natural homomorphism $\phi : G_2 \to G_2/N$, in other words, $N = \phi^{-1}\{0\}$. So

$$\pi^{-1}N = \pi^{-1}\phi^{-1}\{0\} = (\phi \circ \pi)^{-1}\{0\},$$

that is, $\pi^{-1}N$ is the kernel of the homomorphism $\phi \circ \pi : G_1 \to G_2/N$. Thus $\pi^{-1}N$ is a normal subgroup of G_1.

We are now ready to assemble these ingredients into the following proposition, from which the abelian quotient theorem follows immediately. Two pieces of terminology will be used. An element h of G is called a *normal generator* of G if the only normal subgroup of G containing h is G itself. A group is called *simple* if it has precisely two normal subgroups—inevitably, the whole group and the one-element subgroup.

PROPOSITION 4.4. *Let G be a group.*
(a) If $D(G) \leqslant 2|G|$ then G has a normal generator.
(b) If G has a normal generator then any abelian quotient of G is cyclic.

Proof. Suppose that $D(G) \leqslant 2|G|$. Then by lemma 4.3, the mean over all $g \in G$ of

$$\nu(g) := |\{\text{normal subgroups of } G \text{ containing } g\}|$$

is less than or equal to 2. If G is not simple or trivial then $\nu(1_G) \geqslant 3$ (where 1_G is the identity element of G); so for the mean to be less than or equal to 2, there must be some $h \in G$ for which $\nu(h) = 1$—and this says exactly that h is a normal generator of G. On the other hand, if G is simple then any nonidentity element of G is a normal generator, and if G is trivial then 1_G is a normal generator. So (a) is proved in all cases.

For part (b), let A be an abelian quotient of G with $\pi : G \to A$ a surjective homomorphism, and let h be a normal generator of G. Then $\pi(h)$ is a normal generator of A: for if $K \trianglelefteq A$ and $\pi(h) \in K$ then $\pi^{-1}K$ is a normal subgroup of G containing h, so $\pi^{-1}K = G$; and since π is surjective, this means that $K = A$. But A is abelian, so all subgroups are normal, so the fact that $\pi(h)$ is a normal generator of A says that the only subgroup of A containing $\pi(h)$ is A itself. And this in turn says exactly that the cyclic subgroup generated by $\pi(h)$ is A itself. □

5. The Abelian Quotient Theorem: Proof by Prime-Index Subgroups

This last section is devoted to a second proof of the abelian quotient theorem, 4.1. This time, the proof reveals something about the normal subgroup structure of a perfect group G: namely, that G has at most one normal subgroup of each prime index (5.2(a)). It is a corollary of this that any abelian quotient of G is cyclic.

This section assumes some more sophisticated group theory than the last.

LEMMA 5.1. *Let G be a group and p a prime: then the number of normal subgroups of G with index p is*

$$\frac{p^r - 1}{p - 1} = 1 + p + \cdots + p^{r-1},$$

for some $r \geqslant 0$.

(Note that 'usually' $r = 0$, in which case both sides of the equation evaluate to zero.)

Proof. For this proof we write the cyclic group of order p additively, as $\mathbb{Z}/p\mathbb{Z}$. We also write $\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ for the set of all homomorphisms $G \to \mathbb{Z}/p\mathbb{Z}$, and $\operatorname{Aut}(\mathbb{Z}/p\mathbb{Z})$ for the set of all automorphisms of the group $\mathbb{Z}/p\mathbb{Z}$ (that is, invertible homomorphisms $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$).

The key observation is that a normal subgroup of G of index p is just the kernel of a surjection from G to $\mathbb{Z}/p\mathbb{Z}$.

All but one element of $\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ is surjective, and the remaining one is trivial. Two surjections $\pi, \phi : G \to \mathbb{Z}/p\mathbb{Z}$ have the same kernel if and only if $\pi = \alpha \circ \phi$ for some $\alpha \in \operatorname{Aut}(\mathbb{Z}/p\mathbb{Z})$; moreover, if such an α exists for π and ϕ then it is unique. So the nontrivial elements of $\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ have

$$\frac{|\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})| - 1}{|\operatorname{Aut}(\mathbb{Z}/p\mathbb{Z})|}$$

different kernels between them. In other words, there are this many index-p normal subgroups of G. We now just have to evaluate $|\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})|$ and $|\operatorname{Aut}(\mathbb{Z}/p\mathbb{Z})|$.

Firstly, $\mathbb{Z}/p\mathbb{Z}$ is cyclic with $p - 1$ generators, so $|\operatorname{Aut}(\mathbb{Z}/p\mathbb{Z})| = p - 1$.

Secondly, $\mathbb{Z}/p\mathbb{Z}$ is abelian, so $\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ forms an abelian group under pointwise addition. Each element has order 1 or p, so $\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ can be given scalar multiplication over the field $\mathbb{Z}/p\mathbb{Z}$, and thus becomes a finite vector space over $\mathbb{Z}/p\mathbb{Z}$. This vector space has a dimension $r \geqslant 0$, and then $|\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})| = p^r$. (Alternatively, Cauchy's Theorem gives this result.) This completes the proof. □

Let us temporarily call a group G *tight* if for each prime p, G has at most one normal subgroup of index p. Putting together the three parts of the following proposition gives us our second proof of the abelian quotient theorem.

PROPOSITION 5.2.
(a) A group G with $D(G) \leqslant 2|G|$ is tight.
(b) A quotient of a tight group is tight.
(c) A tight abelian group is cyclic.

Proof. For part (a), note that for each prime p we have

$$2|G| \geqslant D(G) \geqslant |G| + \frac{p^r - 1}{p - 1} \cdot \frac{|G|}{p},$$

where r is as in lemma 5.1. If $r \geqslant 2$ then

$$\frac{p^r - 1}{p - 1} \cdot \frac{|G|}{p} \geqslant (p + 1) \cdot \frac{|G|}{p} > |G|,$$

giving a contradiction. Thus r is 0 or 1, and so $(p^r - 1)/(p - 1)$ is 0 or 1.

For part (b), let $\pi : G_1 \to G_2$ be a surjective homomorphism. If N and N' are distinct normal subgroups of G_2 with index p, then $\pi^{-1}N$ and $\pi^{-1}N'$ are distinct normal subgroups of G_1 with index p.

To prove (c) we invoke the classification theorem for finite abelian groups, which tells us that for any abelian group A there exist primes $p_1, \ldots, p_n$ and numbers $t_1, \ldots, t_n \geqslant 1$ such that

$$A \cong C_{p_1^{t_1}} \times \cdots \times C_{p_n^{t_n}}.$$

Suppose that $p_i = p_j$ ($= p$, say) for some $i \neq j$. Then, since $t_i \geqslant 1$, $C_{p^{t_i}}$ has a (normal) subgroup N_i of index p; and similarly $C_{p^{t_j}}$. Hence $N_i \times C_{p^{t_j}}$ and $C_{p^{t_i}} \times N_j$ are distinct index-p subgroups of $C_{p^{t_i}} \times C_{p^{t_j}}$, and $C_{p^{t_i}} \times C_{p^{t_j}}$ is not tight. Since $C_{p^{t_i}} \times C_{p^{t_j}}$ is a quotient of A, part (b) implies that A is not tight either. Thus if A is tight then all the p_k's are distinct, so that

$$A \cong C_{p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n}}.$$

$\square$

There are still other lines of proof for the abelian quotient theorem. In part (b) of the proposition, the fact that p was prime was quite irrelevant, and in just the same manner we can prove that

$$\frac{D(G_1)}{|G_1|} \geqslant \frac{D(G_2)}{|G_2|}$$

whenever G_2 is a quotient of G_1. (If $\pi : G_1 \to G_2$ is the quotient map, with kernel of order k, then a normal subgroup N of G_2 gives rise to a normal subgroup $\pi^{-1}N$ of G_1 of order $k|N|$.) Thus if G is a group with $D(G) \leqslant 2|G|$ and A is an abelian quotient of G then $D(A) \leqslant 2|A|$. So we have reduced the abelian quotient theorem to the abelian case: if A is abelian and $D(A) \leqslant 2|A|$ then A is cyclic. Certainly this is provable by methods derived from one of the two proofs of the general case, but other approaches exist; I leave that for the reader.

Further Thoughts

We finish with some general speculative thoughts, roughly in order of the material above.

The chosen definition of the function D, and therefore of perfect group, is one amongst many candidates. We defined D to be the sum of the orders of the normal subgroups, but we could change 'normal subgroups' to 'subgroups', 'characteristic subgroups', 'subnormal subgroups', ..., or we could define D to be the sum of the *indices* of the normal subgroups, etc. In all cases we preserve the identity $D(C_n) = D(n)$, but only in some of them does D remain multiplicative (a feature we probably like).

More abstractly, this article was about lifting the classical function

$$D : \{\text{numbers} \longrightarrow \{\text{numbers}\}$$

to a function

$$D : \{\text{groups}\} \longrightarrow \{\text{numbers}\}.$$

We might consider it natural to go the whole hog and create a function assigning not just a number, but some kind of algebraic structure, to each group G. I do not know of any very useful way to do this.

In number theory there is a whole body of work on multiplicative functions of integers, which include the number-of-divisors function, the sum-of-divisors function, the Euler function ϕ, and the Möbius function μ. In the world of groups we have at least the beginning of an analogue. For let F be a multiplicative function from groups to numbers: then just as in corollary 3.2, the function $F' : G \mapsto \sum_{N \trianglelefteq G} F(N)$ is multiplicative. For instance, if F is the function with constant value 1 then F' gives the number of normal subgroups of a group, and is multiplicative.

The abelian quotient theorem says that if $D(G) \leqslant 2|G|$ then G has some special property expressible in standard group-theoretic terms. We can prove this in at least two ways, but it seems rather more challenging to prove something in the other direction: that if $D(G)$ is 'too big' then G has a certain form.

Finally, we can make various conjectures on perfect groups, based on the skimpy evidence above: for instance, 'there are no odd-order perfect groups', or 'there are infinitely many nonabelian perfect groups'. Example 2.4, on the dihedral groups, tells us that classifying the even-order perfect groups is at least as hard as determining whether there are any odd perfect numbers. Clearly such problems are unlikely to be easy to solve.

Problems Drive 1996

Gareth and Emma McCaughan

The final score will the length of the vector whose components are your scores on the individual problems. In some cases only approximate answers may be possible. In others they may be unnecessary but acceptable. In others they may be of no use at all. Use your judgement.

1 Below are two lists. The first contains mathematicians, the second more-or-less mathematical statements. (They may be theorems, conjectures, hypotheses or anything else.) Pair off the statements with the mathematicians with whose names they are usually associated. When you have paired them all off, one statement will remain. Give its usual name. We reserve the right to have obfuscated some of the statements.

1	Apollonius	7	Newton
2	Archimedes	8	Riemann
3	Fermat	9	Minkowski
4	Gauss	10	Waring
5	Gödel	11	Wolstenholme
6	Hardy	12	Zassenhaus

a Let $p > 3$ be a prime. Then the numerator of $\sum_{k=1}^{p-1} \frac{1}{k}$ is a multiple of p^2.

b $n = \Delta + \Delta + \Delta$.

c $n = x_1^k + x_2^k + \cdots + x_r^k$.

d Any conic section is the locus of a point which moves so that the ratio of its distance from a fixed point to its distance to a fixed line is constant.

e To every ω-consistent recursive class κ of formulae there correspond recursive class-signs r, such that neither v Gen r nor Neg (v Gen r) belongs to Flg(κ) (where v is the free variable of r).

f $(A_1 \cap A_2)C_1/(A_1 \cap C_2)C_1 \cong (A_2 \cap A_1)C_2/(A_2 \cap C_1)C_2$.

g $223/71 < \pi < 22/7$.

h The gene frequencies in a large population remain constant from generation to generation.

i
$$\begin{aligned}
&\forall a \forall b \forall c((\,(\exists d \exists e\ d \in a \,\&\, e \in a \,\&\, \forall f(f \in a \Rightarrow (f = d \vee f = e)))\\
&\&(\exists g \exists h\ g \in b \,\&\, h \in b \,\&\, \forall i(i \in b \Rightarrow (i = g \vee i = h)))\\
&\&(\forall j((\forall k(k \in j \Rightarrow k \in a \,\&\, k \in b)) \Rightarrow (\forall l(l \in a \,\&\, l \in b \Rightarrow l \in j))))\\
&\&(\forall m(m \in c \Longleftrightarrow m \in a \vee m \in b))\\
&\Rightarrow (\exists n \exists o \exists p \exists q\ n \in c \,\&\, o \in c \,\&\, p \in c \,\&\, q \in c \,\&\\
&\qquad \forall r(r \in c \Rightarrow (r = n \vee r = o \vee r = p \vee r = q))).
\end{aligned}$$

j If A is a convex region in $\mathbb{R}^n$, symmetrical about O and of volume greater than 2^n then $\#(A \cap \mathbb{Z}^n) > 2$.

k $(1+x)^n = 1 + nx + n(n-1)/2x^2 + \cdots$.

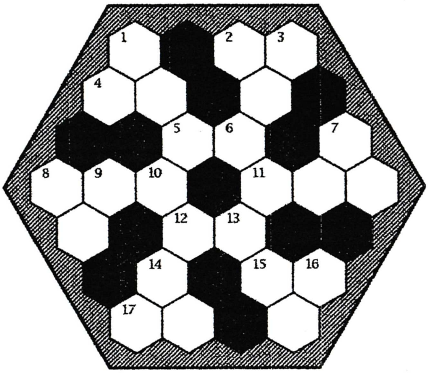
l Light always travels by the fastest route available.

m $ds^2 = \sum g_{ij} x_i x_j$.

2 Give the next two elements of each of the following sequences. Give also a brief statement of what each sequence is.

a 2, 4, 6, 8, 10, 11, 12, 13, ___, ___, ...
b 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, ___, ___, ...
c 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, ___, ___, ...
d 10, 37, 31, 28, 30, 6, 16, ___, ___, ...

3 Solve the crossnumber in the grid provided. All answers are in base 10, and no answer begins with a zero. You are given the following information, where "*n*x" indicates a number starting in the cell *n* and going in direction **x**.



5SW, 10SE, 17E are a pythagorean triple
7SW, 7SE are triangular numbers
1SE is a Fibonacci number
12SW is a palindromic multiple of 7
8E, 15E, 13SE, 16SW are (non-trivial) powers of integers
6SE=11SW+12E
1SW+4E=8SE+9SW
3SW is a perfect number
11E is twice a prime number
2E is twice a prime number
2SE is thrice a prime number

4 My pet mouse died in a tragic maze-running accident some time last year, and I am now experimenting with even less expensive animals. My current project involves a tame worm named Ethelred, whom I have trained to navigate a grid in the following fashion:

Each square of the grid has 1, 2, 3 or 4 bumps in it. At any given moment, Ethelred's head is in one square and his tail is in one of its four neighbours. Once a minute, Ethelred counts the bumps in those two squares; suppose there are m in his tail's square and n in his head's square. Then he turns his head through $n-m$ right angles anticlockwise, moves his head one square forward and puts his tail where his head used to be.

Initially I put Ethelred's head in at one square of my grid, leaving his tail outside. One minute later, he moves forward one square. I then wait and see if his head ever comes out at the edge of the grid; if it does, he gets terribly confused and I remove him.

Recently the company that makes grids for me has started making covered grids. I can no longer see how many bumps there are in each square, or what Ethelred does inside the grid. The diagram on the answer sheet shows a grid they sent to me yesterday; I've been trying to work out what's inside it.

I put Ethelred in at E, 2 minutes later his head emerged at C.
Then I put him in at A, and he came out at K 5 minutes later.
Then I put him in at P, and he came out at B 2 minutes later.
Then I put him in at J, and he came out at L 5 minutes later;
when I put him back in at L it took a further 11 minutes before
he emerged at B.
A couple of hours ago I put him in at B. I think this was a mistake,
as I haven't seen him since.

I know only two things about the grid: firstly, the top-left square (with exits A and P) has exactly one bump in it; secondly, there are at least three squares with each possible number of bumps. Please fill in the rest for me.

	A	B	C	D	
P	**1**				E
O					F
N					G
M					H
	L	K	J	I	

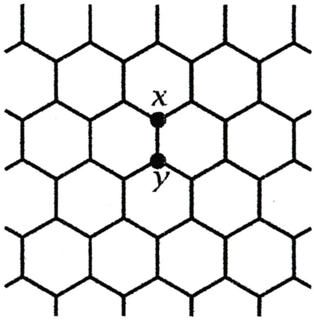
5 The 6 × 10 grid on the answer sheet is actually occupied by a complete set of pentominoes. Each pentomino has five digits on it, all different; the sum of the digits on each pentonimo is the same.

Mark the edges of the pentominoes on the grid.

9	8	3	4	7	7	3	1	7	9
1	3	3	4	6	4	9	5	7	2
2	4	5	4	9	9	5	1	9	6
5	6	7	7	6	1	5	1	2	6
4	6	5	3	2	2	1	8	2	6
8	6	4	5	9	1	9	1	9	7

(A pentomino is a planar shape made up of five equal-sized squares, joined along their boundaries. Two pentominoes are considered to be the same if they are equivalent under an isometry of the plane. There are 12 different pentominoes. A well-known puzzle is to fit them all into a 6×10 box without overlapping.)

6 Each edge of the infinite hexagonal mesh of which a portion is shown below has unit resistance. What is the overall resistance between vertices x and y?



7 Find all values of t in $(0, \pi)$ for which

$$\cos t + \sin t + \tan t + \sec t + \operatorname{cosec} t + \cot t = 6.4.$$

8 Compute as accurately as you can,

$$\int_0^1 \cos^{100} t \, dt.$$

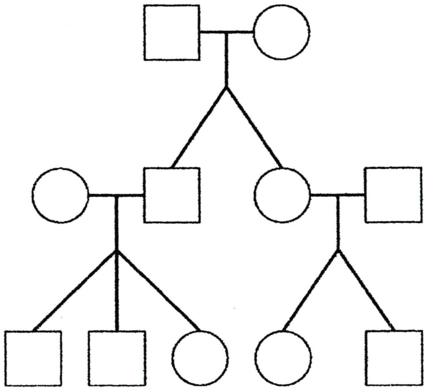
9 Bert, Gert and Kurt are playing a Ping-Pang-Pong match. Don't worry if you aren't familiar with the rules of this excellent game: all you need to know is that every game has a single winner, who scores one point, and that in order to win a match you need to have (i) at least 9 points and (ii) at least 2 points more than either of your opponents.

This match is very exciting. The three of them are evenly matched, and since Ping-Pang-Pong is a game of pure luck anyway they clearly each have the same chance of winning each game. At the moment the score stands at a nail-biting $9-9-9$. What is the probability that Bert will win and Gert come second, with scores $14-12-11$?

10 Suprisingly enough, the letters in the following addition sum represent the digits $1, \ldots, 9$. What number is represnted by "CAYLEY"?

```
 CAUCHY
 CAUCHY
 ------
 EUCLID
```

11 In the family tree below, square boxes indicate male family members and round boxes female family members. Each set of siblings is in descending order of age from left to right across the page. Exactly five of the people represented on the tree are mathematicians. Unusually for families appearing in this sort of problem, they all always tell the truth.



Last Christmas they were assembled together (and no one not on the family tree was there), and they made the following statements. Of course, none of them would be so rude as to mention anyone who wasn't present.

A: My father is a mathematician.
B: So is mine.
C: And mine.
D: And mine.
E: Mine too.
F: Neither of my parents is a mathematician, but my brother is one.
G: Only one of us females is a mathematician, but it's not me.
H: Well, I've given birth to two mathematicians, no more and no less.
J: My brother-in-law is a mathematician.
C: My husband isn't. [Lucky *C*—Ed.]
D: Exactly one of my cousins is.
G: *B* is my granddaughter.

E is the youngest male mathematician in the family, followed by *K*; *I* is the oldest. [Surely 'I *am* the oldest'?—Ed.] You may take *F*'s statement to imply that *F* has exactly one brother.

Fill in each space in the family tree with the corresponding letter, and indicate in some suitable manner which family members are mathematicians and which are not. There are two possible solutions.

12 Az'glqssk the alien lives on a platonic planet whose shape is that of a perfect regular dodecahedron. Being extremely xenophobic, she checks every day that no-one else has arrived on her planet. In order to do this, she has to travel a closed path (from her house, to her house) with the property that every point on the planet's surface is visible from some point of the path. If the edge-length of the dodecahedron is one unit, what is the length of the shortest path she can take?

(Az'glqssk is affraid of heights as well as of strangers, and her path must therefore be confined to the surface of the planet.)

Obituary Notice: Cedric A. B. Smith

The Archimedeans are sorry to learn of the death of our longest standing contributor to *Eureka* on the 11th February 2002, aged 84.

Cedric Smith was a member of the Trinity Mathematical Society as an undergraduate in the 1930s at the time when the Archimedeans had just been founded. He was involved with an undergraduate group which successfully solved the problem of "squaring the square"; subdividing a square into squares of distinct sizes, which they achieved by cunning use of an equivalent electrical network representation, which was subsequently described in Eureka 34 in 1971.

This 55th issue of Eureka contains his final two contributions to this journal under the same pseudonym which was used in the squares article, *Blanche Descartes.* This editor shall particularly remember a telephone conversation when Professor Smith was requesting permission to reproduce an article on the three coin problem, from an early issue of Eureka. He explained that he had already taken the necessary steps in contacting the author, Blanche Descartes, and she was quite happy to grant the necessary permission. It did not dawn upon me at the time that they might be one and the same person.

The Intersecting Chords Theorem

Colin Bell

1. Introduction

The classical Intersecting Chords Theorem is well known: if AB and PQ are chords of a circle which intersect at O, then $AO.OB = PO.OQ$. In this paper we prove a natural analogue of this result: that the ratio $PO.OQ/AO.OB$ is bounded above and below on convex C^2 curves of bounded curvature. The question was postulated by Alan Beardon, who proved a related result: given the same setup, but with no curvature or smoothness conditions on the curve, and with the lengths of the two chords bounded below, then the ratio is bounded above and below [1].

The result is the following:

THEOREM 1. *Let $\mathcal{D}$ be the set of all convex open domains whose boundary is an admissible curve with curvature bounded above and below by positive numbers K and k respectively, and D be a member of $\mathcal{D}$. Let AB and PQ be chords of D which intersect at O in D. Then*

$$\frac{k}{K} \leqslant \frac{PO.OQ}{AO.OB} \leqslant \frac{K}{k}.$$

For a given K and k, this bound is the best possible. Furthermore, the inequalities are strict unless D is a disc (where $K = k$).

An *admissible* curve is any for which our extended version of the Blaschke Rolling Theorem (Theorem 3) holds. In particular, it includes the class of those curves which are C^2 except possibly at a finite number of points, where it is C^1, and where in addition, the curvature at all C^2 points, and the limits of the curvature as the C^1 points are approached from either side, are bounded above and below.

We shall only prove the upper bound; the lower follows by symmetry. We shall generally think in terms of the radii of circles rather than curvatures and define

$$R = \frac{1}{k}, \quad r = \frac{1}{K};$$

hence $0 < k \leqslant K$ and $0 < r \leqslant R$.

To prove the result, we first note the following general principle which appears at several points in different versions during the proof: given a domain D with the chords AB and PQ fixed, if we have a domain $E \supset D$, with AB also a chord of E, and extend PQ to meet ∂E at P' and Q', then

$$\frac{P'O.OQ'}{AO.OB} \geqslant \frac{PO.OQ}{AO.OB}.$$

In particular, this means that provided we can find a suitable E for any choice of D and a pair of chords PQ, AB, and we know the theorem for E, we can derive it for D. In section 2, we will define a family $\mathcal{E}$ and show that it always has an appropriate E in it. Then in section 3 we prove what we need of the result for $\mathcal{E}$, which gives us the main part of Theorem 1.

We then show in Section 4 that our result is best possible, and finally in Section 5 show that if the curvature is unbounded in either direction then no such bound exists.

2. Curvature

The curvature $k(s)$ of a C^2 curve is defined to be

$$k(s) = \frac{d\theta}{ds},$$

where s is arc length, and θ is the angle the tangent to the curve at that point makes with a fixed line. The curve needs to be C^2 for this (in particular $d\theta$) to be defined.

We shall need the following result: the argument is taken from [2], part of Theorem 2-14 and the preceding discussion.

Let f and g be two curves which are tangent at a point. We may assume this is the origin and the 0 point of both curves, and the tangent is the x-axis. Because the function θ is continuous for a C^2 curve, there is an interval along the x-axis for which both curves can be considered as functions $F(x)$ and $G(x)$: the interval being that on which $|\theta_f|, |\theta_g| < \pi/2$. We define f to be *above* g if $F(x) \geqslant G(x)$ with equality only at 0. We now have the following:

LEMMA 2. *Let f and g be as above. If $k_f(0) > k_g(0)$, f is above g at 0.*

We shall also need the following result, which is a version of Blaschke's Rolling Theorem given in [3]:

THEOREM 3. *Let C and $\bar{C}$ be two closed C^2 convex curves which are tangent to each other at some point with the same unit normal there and for $P \in C$ and $\bar{P} \in \bar{C}$, $k(P) \leqslant \bar{k}(\bar{P})$ if the unit tangents at P and $\bar{P}$ are equal. Then $\bar{C}$ lies entirely within the closed convex set bounded by C.*

The proof works equally well for curves which are only C^1 at a finite set of points: the only change we need to make is that when one of the C^1 points is being considered, the parts of the curve on either side of it should be thought of as separate C^2 curves, but both with the correct curvature properties.

We shall only need the result in the following cases:

LEMMA 3.

(i) *Let D be a convex smooth curve with curvature bounded below by k. Then D lies inside any circle of radius $1/k$ tangent to D at some point.*

(ii) *Let D be a convex smooth curve with curvature bounded above by K. Then D lies outside any circle of radius $1/K$ tangent to D.*

3. Construction of $\mathcal{E}$

For $R \geqslant r > 0$ and $0 \leqslant \beta \leqslant \pi/2$, we define $E(R,r,\beta)$ as follows (see Figure 1). Let

$$x_0 = (R-r)\sin\beta, \quad y_0 = (R-r)\cos\beta.$$

Define Γ_2 and Γ_4 to be circles of radius r centred at $(x_0,0)$ and $(-x_0,0)$, Γ_1 and Γ_3 to be circles of radius R centred at $(0,-y_0)$ and $(0,y_0)$, and C_i to be the centre of Γ_i. Then an easy calculation shows that Γ_1 and Γ_2 are mutually tangent at a point $I_1 = (R\sin\beta, r\cos\beta)$. (Note that such a point must be on the extended line C_1C_2.) The same holds by symmetry for the remaining circles, and hence we can define $E(R,r,\beta)$ to be the domain bounded by the outer arcs of the four circles. We define $\mathcal{E}$ to be the set of all such $E(R,r,\beta)$.

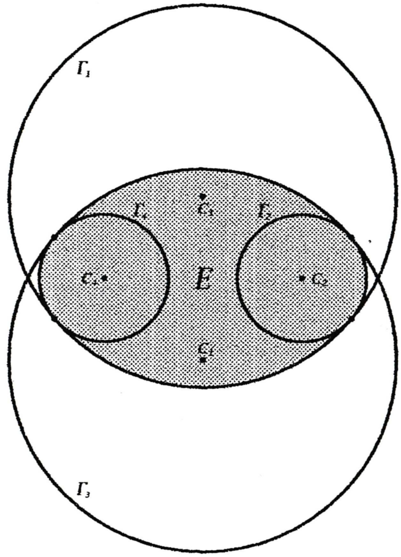


Figure 1

Now start with a domain D and two chords AB and PQ meeting at O in D. Define γ_1, centred at C_1, to be the circle of radius R tangent to D at A, and γ_3, centred at C_3, that of radius R tangent at B. By part (i) of Lemma 3, both γ_1 and γ_3 contain D. Define L to be their intersection, a lens-shaped region. If γ_1 and γ_3 coincide, then L is a circle, and so we immediately have that $PO.OQ/AO.OB \leqslant 1$, and we shall disregard this case from all further discussion. Otherwise, orientate the picture such that the centres of γ_1 and γ_3 are on the y-axis, equidistant from the origin.

Note that for an appropriate β, γ_1 and γ_3 are the same as the Γ_1 and Γ_3 in the construction of $E(R,r,\beta)$, and it is this domain (which we shall refer to just as E)

that we shall try to fit D inside. We define E' to be the subset of L which lie on or inside some circle of radius r whose interior lies inside L. We know D is contained in E', since by the second part of Lemma 3, we have a circle inside D touching every point of the boundary, and by convexity the interior of D is in E' as well. So if we can show that $E' \subset E$, we're done. The proof, although elementary, is somewhat messy, and is omitted. $\square$

In fact $E' = E$: this fact is self-evident if a diagram is drawn.

4. Proof of the Theorem for $\mathcal{E}$

We will fix a particular $E(R, r, \beta)$ and just call it E. We note the following immediately: E is symmetric in both co-ordinate axes. We only need to prove the theorem for E in the case when A and B lie on $\Gamma_1 \cup \Gamma_3$: since this is forced by the construction we used in Section 2. We shall always have O in the $\Gamma_1\Gamma_2$ quadrant, and have A on Γ_1.

We can immediately deal with the case when A and B are both on Γ_1, since then one of P or Q (we may assume the former), is also on Γ_1. Extend OQ to meet Γ_1 at R. The ICT for Γ_1 says that $OP.OQ/OA.OB \leqslant OP.OR/OA.OB = 1$, which gives us the result we want here. So we shall assume B is on Γ_3.

We now fix O and consider what choices of the other points maximise the ratio we can get: firstly AB and then PQ.

LEMMA 5. *For a fixed O, with A on $\partial E \cap \Gamma_1$ and B on $\partial E \cap \Gamma_3$ such that O is on AB, $AO.OB$ is minimal when AB is parallel to the y-axis.*

PROOF. (See Figure 2) Let A,B be an arbitrary choice satisfying the conditions, and R,S be the equivalent points on the vertical through O. We require $OR.OS \leqslant OA.OB$.

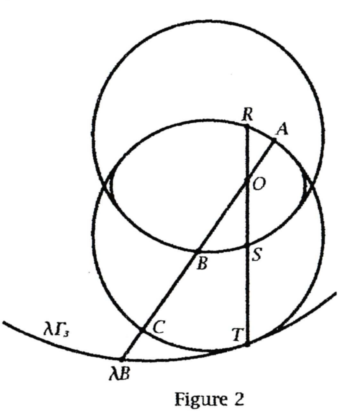


Figure 2

Extend OB to meet Γ_1 again at C; T is the equivalent point on OS. By the ICT applied to Γ_1, $OA.OC = OR.OT$, so our desired result is equivalent to $OT/OS \geqslant OC/OB$. Let $\lambda = OT/OS$, and expand Γ_3 by a factor of λ centred at O. Since OT is parallel to the line joining the centres of the two circles and they have the same radius, the angles it makes with the tangents at S and T are the same, and hence $\lambda\Gamma_3$ is tangent to Γ_1 at T. Consider the action of this expansion on B: it becomes a point on $\lambda\Gamma_3$ which is outside Γ_1 except at T (two tangent circles of different sizes), and hence $\lambda OB > OC$ which gives us the inequality we want. □

Lemma 6. *For a fixed O, with P and Q on ∂E such that O is on PQ, $OP.OQ$ is maximal when PQ is parallel to the x-axis.*

Proof. (See Figure 3) Let PQ be arbitrary satisfying the conditions, and RS be the equivalent points on the line through O parallel to the axis. If we can find a circle for which RS is a chord, and which contains E then we are done, since we extend OP to P' and OQ to Q' on this circle, and then $OP.OQ < OP'.OQ' = OR.OS$ by the ICT applied to it.

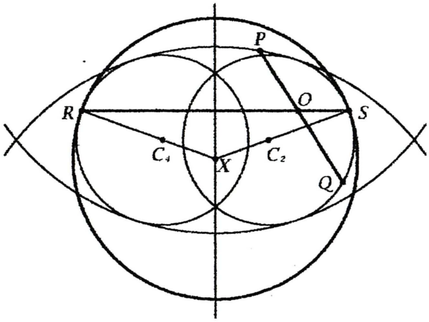


Figure 3

If R and S are on Γ_1, then just take Γ_1 as the circle. If not, then consider the diameters of Γ_2 and Γ_4 at R and S. Their intersection point X (extending them if necessary) is on the y-axis by symmetry. Draw the circle centered there, and passing through R, and call it C. It also passes through S (again by symmetry). To show that it contains E, we compare the distance to R with the distance to other points on ∂E. We use the fact that for an arbitrary circle C and a point X, the distance from X to points on C has one minimum and one maximum, attained at the two ends of the diameter of C which passes through X, the minimum being the point on the same side as the centre, and the maximum being the point on the opposite side.

For a point W on Γ_2, we obtain immediately that $XW \leqslant XR$. If W lies on Γ_1, then we know the intersection with the y-axis is either a minimum or a maximum: since X lies above C_1 (since I_1C_1 and RX cross at C_2) we know it is a minimum, so we have $XW \leqslant XI_1 \leqslant XR$, as required. So this circle contains E as required. □

Having established that the only cases we need to consider are those with AB vertical and PQ horizontal, we can now calculate the ratio $PO.OQ/AO.OB$ explicitly. The argument splits into two cases. The first is where P and Q are both on Γ_1. We first note that as for the same reasons as Lemma 5, $OP.OQ/OA.OB = OC/OB$. P and Q both have y-coordinates $\geqslant r\cos\beta$ (they are above I_1,I_4), hence the same condition applies to O, and that of B is $\leqslant -r\cos\beta$, giving $OB = (2r+\mu)\cos\beta$, with

$\mu \geqslant 0$. $BC = 2(R - r)\cos\beta$ (since BC is parallel to the line of centres). Putting this together we get $OC/OB = (2R + \mu)/(2r + \mu) < R/r$ as required.

The other case has P and Q on Γ_2 and Γ_4. We let A and P be generic points on Γ_1 and Γ_2 respectively and with the assumptions we have about AB and PQ being parallel to the axes, we have that A and B are $(\pm R\sin\alpha, R\cos\alpha - (R - r)\cos\beta)$, and P and Q are $(r\sin\gamma + (R - r)\sin\beta, \pm r\cos\gamma)$ with $\alpha \leqslant \beta \leqslant \gamma$. So we have

$$\frac{PO.OQ}{AO.OB} = \frac{(r\sin\gamma + (R - r)\sin\beta + R\sin\alpha)(r\sin\gamma + (R - r)\sin\beta - R\sin\alpha)}{(R\cos\alpha - (R - r)\cos\beta - r\cos\gamma)(R\cos\alpha - (R - r)\cos\beta + r\cos\gamma)}.$$

We want this to be less than R/r, which is equivalent to $R.AO.OB - r.PO.OQ \geqslant 0$. Dividing both sides of this through by r^3, writing λ for R/r, and expressing the terms as differences of squares we obtain

$$R.AO.OB - r.PO.OQ$$

$$= \lambda((\lambda\cos\alpha - (\lambda - 1)\cos^2\beta) - \cos^2\gamma) - (\sin\gamma + (\lambda - 1)\sin^2\beta^2) + \lambda^2\sin^2\alpha$$

This can be simplified to

$$(\lambda - 1)(\lambda^2(\cos\alpha - \cos\beta)^2 + (\sin\beta - \sin\gamma)^2) \qquad (*)$$

which is positive as required.

Having now proved the result (or at least the part we need) for $\mathcal{E}$, we can derive it for $\mathcal{D}$ for the reasons given before. The proof of the main part of Theorem 1 is now complete.

5. Optimality

We can now show that the result is the best possible, and that equality occurs only when D is a disc. The latter is easier: if $PO.OQ/AO.OB = R/r$, then $(*) = 0$, so either $\lambda = 1$, which implies D is a disc, or $\alpha = \beta = \gamma$. However in this case A and P coincide, and hence O is on the boundary of D, and hence not in D.

To show it is best possible, we will start in $\mathcal{E}$: consider the cases where $\alpha = \beta$. Here (using the same simplification as before), we have

$$\frac{PO.OQ}{AO.OB} = \frac{(\sin\gamma + (2\lambda - 1)\sin\beta)(\sin\gamma - \sin\beta)}{(\cos^2\beta - \cos^2\gamma)}.$$

This simplifies to

$$\frac{(\sin\gamma + (2\lambda - 1)\sin\beta)}{(\sin\gamma + \sin\beta),}$$

which tends to λ as γ tends to β.

Having shown that for every $E \in \mathcal{E}$ we have a sequence of chord-pairs with the ratio tending to λ, we shall pick a given pair, and find a sequence of domains E'_ϵ with a C^2 boundary, which approximate E in such a way that the $PO.OQ/AO.OB$ is also approximated. We may and shall assume that $\beta + \epsilon < \gamma$. (For ease of notation, the ϵ will be implicit.)

As E is convex, we can consider the curvature of its boundary, $k_E(\theta)$, to be a function of the angle of the unit normal to the positive x-axis: we find that in the first quadrant it is K on $0 \leqslant \theta < \beta$ and k on $\beta < \theta \leqslant \pi/2$, and undefined at β. We can deform this into a continuous function by doing a linear interpolation, and hence define $k_{E'}(\theta)$ to be $K - \epsilon(K-k)$ on $\beta \leqslant \theta \leqslant \beta+\epsilon$ and k_E on the rest of the first quadrant, extending it to all of $[0, 2\pi]$ by symmetry. Let E' be the domain that has boundary with curvature function $k_{E'}$: we know the curve is complete by symmetry, and also well-defined up to an isometry of the plane (Theorem 2-10 of [6]): we shall assume it has the same orientation as E.

We now want to show that the ratio approximates that of E. Let A' be the point on E' equivalent to A on E (in the sense that the tangents to the two curves at the two points are parallel), and similarly for the other points. O' is the intersection of $A'B'$ and $P'Q'$. Since AB and PQ are purely vertical and horizontal, respectively, and the same properties carry over to $A'B'$ and $P'Q'$, the distance AO is just the difference in y-coordinates between A and P. For an arbitrary curve, this is just

$$\left|\int_{s_1}^{s_2} \sin\theta\, ds\right| = \left|\int_{\theta_1}^{\theta_2} \frac{\sin\theta}{k(\theta)}\, d\theta\right|,$$

by the definition of curvature, where θ is the angle of the tangent to the positive horizontal.

So we have that

$$AO = \left|\int_{\beta}^{\gamma} \frac{\sin\theta}{K}\, d\theta\right|$$

(which is equal to $r\cos\beta - r\cos\gamma$ since $r = 1/K$), and similarly

$$A'O' = \left|\int_{\beta}^{\beta+\epsilon} \frac{\sin\theta}{k_L(\theta)}\, d\theta + \int_{\beta+\epsilon}^{\gamma} \frac{\sin\theta}{K}\, d\theta\right|,$$

which clearly tends to AO as ϵ tends to 0. $B'O' = A'O'$, and we can produce a similar argument for $P'O'$ and $Q'O'$, so $P'O'.O'Q'/A'O'.O'B'$ tends to $PO.OQ/AO.OB$ as ϵ tends to 0.

We thus have a sequence E'_ϵ with $\sup PO.OQ/AO.OB \to \lambda$ as $\epsilon \to 0$ as required. A pertinent question is whether there is *any* domain satisfying the conditions whose sup is actually equal to λ. I conjecture that the answer is no (based on calculations of particular examples) but have little idea how to attempt a proof.

6. Unbounded curvature

We now prove that if our domain is bounded, having a bound on the curvature is necessary for having a bound on the ratio, since we have:

THEOREM 7. *Let D be a simply connected convex bounded domain whose boundary is a convex C^2 curve with curvature either not bounded above or bounded below, then we can find chords AB and PQ of D such that $PO.OQ/AO.OB$ is arbitrarily large or small.*

PROOF. As before, we need only consider the upper limit. We consider two cases:

(i) Curvature not bounded below (Figure 4)

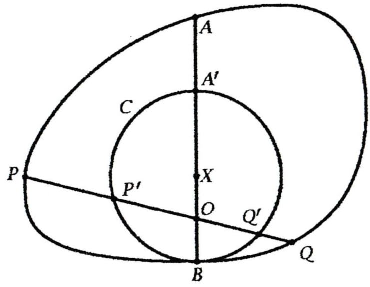


Figure 4

Pick a k. We can find a point B where the curvature is less than k. Draw the circle of curvature $2k$ which is tangent to D at B: call it C and its centre X . Let A be the other intersection point of BX with ∂D. Then there is an interval of ∂D around B with the property that all of it apart from B is outside C, by applying Lemma 2. Pick P and Q in this interval, such that PQ intersects AB at a point O which is nearer B than X is. Now define P',Q' and A' to be the meeting points of OP, OQ and OA with C. The ICT for the circle gives us that $P'O.OQ'$=$A'O.OB$. By construction, $OP > OP'$ and $OQ > OQ'$. Also $A'O$ is at least $1/(2k)$ (the radius of C) since $A'B$ is a diameter. However, we know that D is bounded, and hence has a finite diameter d, which gives an upper bound on OA. So $PO.OQ > P'O.OQ' = A'O.OB' = (A'O/AO)AO.OB$, and hence So we have that $PO.OQ/AO.OB > 2k/d$. Let k tend to 0 and we have the result.

(ii) Curvature not bounded above (Figure 5)

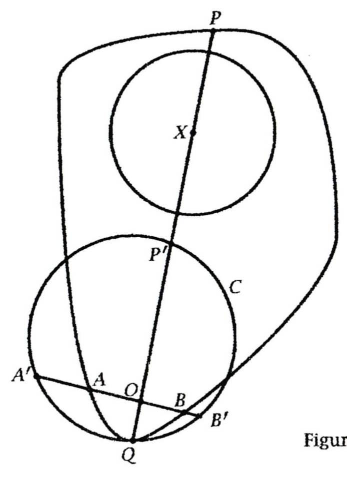


Figure 5

The argument here is much the same. D is open, and hence there is a disc $B(X, d)$ inside it. Again, fix a curvature k, and this time find a point Q where the curvature is greater than k, and draw the circle of curvature $k/2$ tangent to D at Q, again called C. Extend QX to meet D again at P. We can find A and B inside C such that AB meets PQ at O lying on QX, and construct A', B' and P' as before. This time $A'O > AO$ and $OB' > OB$, $OP > d$ and $OP' < 4/k$, the diameter of C, which gives us $PO.OQ/AO.OB > dk/4$. Let k tend to ∞ and we are done. $\square$

7. References

[1] Beardon A.F., *On the dynamics of contractions*, Ergodic Theory & Dynamical Systems.
[2] Guggenheimer H., *Differential Geometry*, McGraw-Hill (1963).
[3] Koutroufiotis, D., *On Blaschke's Rolling Theorems*, Arch. Math. **23** (1972) 655-660.

Une Lettre

Blanche Descartes

Monsieur le rédacteur du journal Euréka!

Monsieur, je voudrais sumettre à votre journal le problème qui suit.

Une probléme élémentaire difficile.

Si a,b et c sont des nombres entiers positifs satisfaisant l'équation

$$a^2 + b^2 = c(1 + ab),$$

démontrer que c est un entier carré: par exemple $a = 2$, $b = 8$, $c = 4 = 2^2$.

On m'a dit que c'est un problème très difficile. Mêmes les mathématiciens les plus habiles du monde ne sont pas réussis à le résoudre.

Néanmoins on m'a dit... qu'il y a une méthode simple afin de le résoudre, si on la sait... Moi je ne la sais pas.

Blanche Descartes,

Cité universitarire, Paris, France

Problems Drive Solutions 1996

Gareth and Emma McCaughan

1 The statements pair off with the mathematicans as given below. The unpaired statement is (i) which is more commonly refered to as $2 + 2 = 4$.

1.	**d**	7.	**k**
2.	**g**	8.	**m**
3.	**l**	9.	**j**
4.	**b**	10.	**c**
5.	**e**	11.	**a**
6.	**h**	12.	**f**

2 Unfortunately you are out of luck with (iv) unless you knew the date of the problems drive in 1996. The completed sequences (with reasons) are as follows:

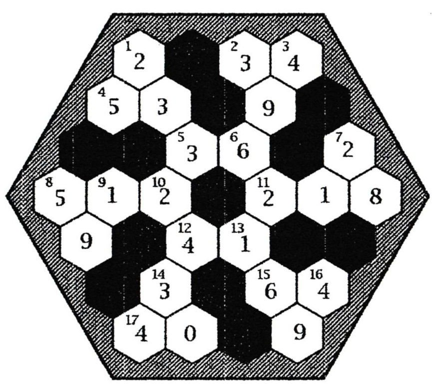
a 2, 4, 6, 8, 10, 11, 12, 13, **14**, **16**, ... (binary non-palindromes)

b 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, **1**, **5**, ... (number of groups of order n)

c 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, **12**, **15**, ... (partitions into distinct parts OR into odd parts)

d 10, 37, 31, 28, 30, 6, 16, **46**, **48**, ... (lottery bonus numbers)

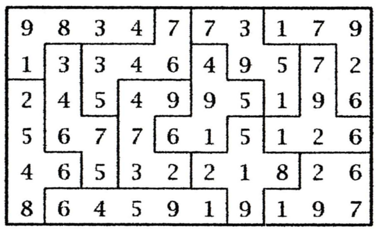
3 The solution to the crossnumber is as follows:



4 The bumps in Ethelred's grid are arranged as follows:

	A	B	C	D	
P	**1**	2	3	4	E
O	1	4	2	3	F
N	2	1	1	2	G
M	2	1	4	3	H
	L	K	J	I	

5 The edges of the pentominoes are marked in on the grid below.



6 The resistance in the network between x and y is $2/3$.

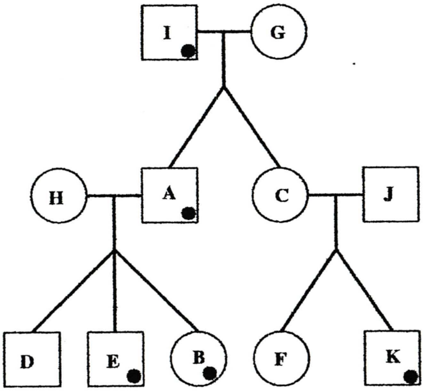
7 The possible values of t are all non-right angles in a 3-4-5 triangle.

8 The value of the integral is 0.12501848...

9 In the Ping-Pang-Pong match the probability that Bert wins, and Gert comes second with scores 14-12-11 is $392/3^{10} = 0.0066385544$. Of course it is taken for granted that everyone should know $3^{10} = 59049$.

10 The letters CAYLEY are 436986.

11 In the family tree below, the mathematicians have been indicated by a dot in the box. Note that there is insufficient information to decided whether D or E is the elder brother.



12 The shortest path which Az'glqssk can take has a total length of $4\sqrt{4} + \sqrt{5} =$ 9.988848.

Problems Drive 1997

Paul Bolchover and Sean Blanchflower

1 Complete the following phrases, where each word has been replaced by its initial letter for example 1 = FP of a CM (1 Fixed Point of a Contraction Mapping).

- **a** 1729 = N of HT
- **b** 4.6692... = R of PD in some DS
- **c** 0.00000007151 = P of W the NL
- **d** $2^{3021376}(2^{3021377} - 1)$ = LKPN
- **e** 3.14159... = LN
- **f** 13 = AP
- **g** $(1+\sqrt{5})/4$ = C of PBF
- **h** $\sqrt{2}/12$ = V of RT of SO
- **i** 23 = HP
- **j** 13 = B of EE
- **k** 6 = RS in FD
- **l** 25 = PBOH

2 Match each of the following equations with the name for their locus:

- **a** $x^{2/3} + y^{2/3} = a^{2/3}$
- **b** $y = x \cot\left(\dfrac{\pi x}{2a}\right)$
- **c** $x = a\sec\theta \quad y = b\tan\theta$
- **d** $\dfrac{dy}{dx} = \pm\dfrac{y}{\sqrt{a^2 - y^2}}$
- **e** $z\bar{a} + \bar{z}a = 2|a|^2$
- **f** $y = \dfrac{8a^3}{x^2 + 4a^2}$
- **g** $x = 2\cos\phi + a\cos 2\phi$; $y = 2\sin\phi - a\sin 2\phi$
- **h** $\left(x^2 + y^2\right)^2 - 2a^2\left(x^2 - y^2\right) = 0$
- **i** $r = -a\dfrac{\cos 2\phi}{\cos\phi}$
- **j** $\alpha z\bar{z} + \bar{\beta}z + \beta\bar{z} + \gamma = 0$
- **k** $(x-a)^2\left(x^2 + y^2\right) = c^2x^2$
- **l** $r = \dfrac{2a}{1 - \cos\theta}$
- **m** $r = a(1 + \cos\theta)$
- **n** $y^2 = \dfrac{x^3}{a - x}$
- **o** $r^{2/3} = \sin^{2/3}\dfrac{\theta}{2} + \cos^{2/3}\dfrac{\theta}{2}$
- **p** $\dfrac{dy}{dx} = \cot\dfrac{\theta}{2}$

- **i** astroid
- **ii** cardioid
- **iii** circle
- **iv** cissoid
- **v** conchoid
- **vi** cycloid
- **vii** deltoid
- **viii** hyperbola
- **ix** lemniscate
- **x** line
- **xi** nephroid
- **xii** parabola
- **xiii** quadratrix
- **xiv** strophoid
- **xv** tractrix
- **xvi** Witch of Agnesi

3 Find the next two terms in the following sequences:

a 100, 200, 300, 301, 302, 303, 304, 309, __, __
b 5, 6, 5, 6, 5, 5, 7, __, __
c 1, 3, 4, 9, 10, 12, 13, 27, 28, __, __
d 0, 2, 3, 4, 5, 5, 7, 6, 6, 7, 11, __, __
e 1, 6, 15, 28, 45, __, __
f 1, 2, 5/2, 17/6, 91/30, __, __

4 Find all four-digit numbers equal to the square of the sum of the two-digit numbers formed by their first two and last two digits (i.e., $ABCD = (AB + CD)^2$.)

5 In the recent Inter-Varsity Real Ludo championship, which ended controversially in a draw, Cambridge's team of Alice, Ben, Carol and David played Oxford's team of Edward, Frances, George and Harriet. The rules state that each team captain must divide their team into two pairs. Each of these pairs plays two matches, one against each of the two pairs from the opposing team. In other words, there are a total of four matches. The first match consists of players 1,2,5 and 6; the second match of players 3,4,7 and 8, the third match of players 1,2,7 and 8 and the fourth match of players 3,4,5 and 6, where 1,2,3 and 4 belong to the team which won the toss. The team is awarded 4 points for each player who comes first, 3 points for second, 2 points for third and 1 point for last. A reporter missed the game, but discovered the following information:

George played two consecutive matches, winning the first but coming last in the second.
Frances got the square of Carol's score.
The result of the last match was in reverse alphabetical order.
Alice consistantly beat her Cambridge team-mate.
In the match where she played Ben, Harriet beat her Oxford team-mate.
Edward beat Ben, but didn't win that match.

Find the results of each match.

6 If (with letters representing digits) ONE+ONE=TWO and ONE+FOUR=FIVE, what are the minimum and maximum values of (FOURTEEN+NINETEEN)?

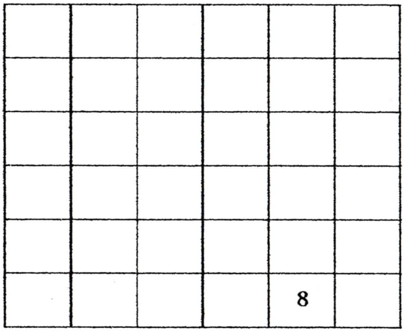
7 A circle is inscribed in quadrilateral ABCD. The sides BC and DA have the same length. The sides AB and CD are parallel, with lengths of 27 and 48 respectively. What is the area of the quadrilateral ABCD?

8 Four unit spheres are stacked to form a tetrahedron. What is the radius of the largest sphere which can be placed in the gap between them?

9 Find and Evaluate to ten decimal places:

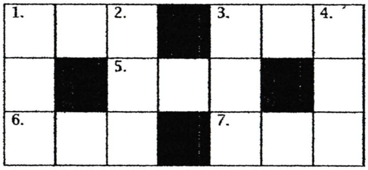
$$\int_0^1 \frac{2x^9(1-x^2)^4}{1+x^4}dx.$$

10 Complete the following magic square, containing the numbers from 1 to 36. The sequences of numbers given all lie in straight lines (horizontally, vertically or diagonally) within the grid.



29, 13, 24, 8, 12
15, 32, 27, 21
20, 11, 2, 18
21, 28, 12
14, 10, 26,11
16, 30, 20
8, 36, 7
20, 4, 13
16, 3, 25

11 In this cross-number, no number has an initial zero. Each clue is the number of factors the answer has, including itself and 1.



Across:	Down:
1. 7	1. 15
3. 15	2. 3
5. 9	3. 3
6. 9	4. 9
7. 15	

12 What is the largest number for which every two consecutive digits form distinct two-digit primes?

Curious Cubes

Blanche Descartes

Alas the news he heard was really bad.
    It made Professor Hardy very sad.
His friend Ramánujan was quite unwell,
    in Evelyn Hospital. (I hate to tell.)
And so he promptly took a taxi down
    towards the southern end of Cambridge town,
and rushing quickly to his friend R's bed,
    he greeted him. To cheer him up he said,
"I thought you possibly might like to hear
    my taxi's numbered 1729. I fear
a number of no interest, I think."
    but yet, before he'd even time to blink,
Ramánujan did answer back, "Oh no!
    it only takes a little thought to show
it's one plus twelve raised to the power of three,
    or cube of nine plus ten cubed, don't you see?
And what is more, no smaller number may
    be written out in more than just one way
as one cube added to another." That is how
    we've always heard the story up till now.
And yet, just when I try to work it out
    I find myself assailed by heavy doubt.
Professor Hardy really is my hero,
    and yet twixt 1729 and zero,
there are some numbers having their displays
    as sums of two cubes in two different ways.
For 728, as you can plainly see,
    its sums of cubes number no less than three.

$$728 = 6^3 + 8^3 = 9^3 + (-1)^3 = 12^3 + (-10)^3.$$

perhaps some clever reader will do more,
    and raise that number three right up to four.

Lectures You May Have Missed

Groups, Rings and Fields
'X is a variable constant.'
'You want to write down the right words, not to make that anything but obvious. . .'
'Just check me, on the front row; you're better at this than me.'

Analysis
'However, this notation is useless.'

Further Analysis
'I've metamorphosed for a few seconds into an applied mathematician.'
'This theorem is remarkable chiefly because it is not due to Cauchy.'
'I hope that the phrase "differentiation is integration" will make some sense after this proof.'

Probability
'Er, I don't know why I said all that. . .' [halfway through a lecture!]

Computational Projects
'However, we recommend that the Macintoshes are used only by experienced users.'

Functional Analysis
'This is not completely trivial. I mean, it's pretty trivial, but it's not completely trivial.'
'The sequence (e_n) is fairly unique.'

Calculus and Methods
'Here's a proof that works most of the time – it's an engineering type of proof.'

Rings, Fields and Modules
'I can assure you that we're not interested in you as individuals.'
'If you're going to be a great mathematician, don't be called Smith.'
'I could say this in 30 seconds if I spoke fast enough.'
'It's a one line proof provided the line is long enough.'

Biological Fluid Mechanics
'Mr. X is studying animal locomotion. He started with bipeds, and has just done quadrupeds. You can consider a quadruped as either two bipeds stuck together, or as a small section of an infinitiped.'

Special Relativity
'You've been using these things for a long time, but because they are the same thing you don't notice they're different.' (Contravariant and Covariant vectors)

General Relativity
'Maybe later I will hold back the c.'

Dynamics
'There's a π and a 2 – that's 2π.'

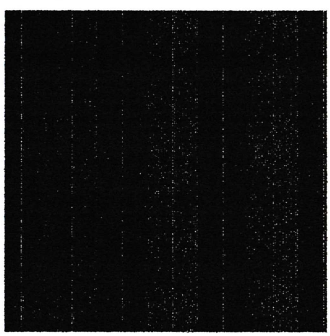
The Origin of the Universe
'Analytic continutation allows you to extend something to something else without changing it really, so for example New Labour are an analytic continutation of old Tories.'

Atmosphere-Ocean Dynamics
'We're all going to start rotating today.'

Stochastic Networks
'By the fundamental theorem of handouts, there exists an error with probability one.'

Seismic Waves
'Big omega is of course extremely small.'

A Peano Space Filling Curve in the Unit Box $[0, 1] \times [0, 1]$



Qarch Solutions

49. Tiling Problem.

The problem was whether a (non-empty) hypercuboid can be tiled by tiles of the shape:

To prove that this is impossible suppose that they tile a hypercuboid. Label the hypercuboid with coordinates, with $(0,0,\ldots,0)$ at one corner and $(a_1,\ldots,a_n)$ at the opposite corner. In each hypercube write $x^{b_1+\cdots+b_n}$ in the cube at $(b_1,\ldots,b_n)$. Then the sum over all the cubes is

$$(1+x+x^2\ldots+x^{a_1})(1+x+x^2\ldots x^{a_2})\ldots(1+x+x^2\ldots x^{a_n})$$
$$=\frac{(1-x^{a_1+1})(1-x^{a_2+1})\ldots(1-x^{a_n+1})}{(1-x)^n}.$$

Now it is easy to see that any tile of the above form covers a multiple of $1+x+x^3+x^5+x^6$ and it is easy to check (e.g., by roots of unity) that this does not divide the above expression.

53. Generalised Borromean Rings Problem.

It is reasonably well known that n unknotted loops of string can be arranged in such a way that, although they cannot be pulled apart, if any one of them is cut then they can all be pulled apart. The following can be deduced: given an up-set U on a set of n loops of string, there exists a configuration of the loops in space such that whenever some subset A of them are cut the remainder can be pulled apart if and only if $A \in U$. [An up-set on a set X is a collection of subsets of X closed under taking supersets.]

56. Fair Dice.

It was asked for which m and n there exist fair n-dimensional m-sided dice. When $n=3$, it turns out that such dice exist if and only if $m \geqslant 4$ is even. If we restrict attention to dice with simplicial faces, then it is known that if m is prime, then either $n=2$, or $n=m-1$ and the die is a regular n-simplex. One way of generating fair (respectively simplicial) dice is to take the dual of the cartesian product of the duals of a pair of fair (respectively simplicial) dice.

77. Integral.

If $t=\sqrt{x^2+y^2}-x$ then the inner integral will have a form

$$(1-x)\int_r^{\infty} t^{b-1}(t+2x)^{b-1}(2+t)^{-c}dt,$$

where $r = \max(0, -2x)$. After changing the order of integrations, the double integral becomes

$$\int_0^\infty t^{b-1}(2+t)^{-c}dt \int_{-\frac{t}{2}}^1 (t+2x)^{b-1}(1-x)^a dx.$$

By making the substitution $x = \frac{1}{2}(u(2+t) - t)$, the inner integral becomes

$$\int_0^1 u^{b-1}(2+t)^{b-1}(1-u)^a(2+t)^{-a-1}(2+t)du = 2^{-a-1}(2+t)^{a+b}B(b, a+1).$$

Hence the integral is equal to

$$2^{2b-c-1}B(b, c-a-2b)B(b, a+1).$$

Carpentry: a fable

Prof. R. Brown

Recently I attended a carpentry course. It was pretty tough.

All the students (or almost all) were eager to learn. The first three weeks we learned to drill holes. We found out about curious kinds of drills, and how to make holes at odd angles. We got pretty good and accurate at drilling holes. The next six weeks were involved in cutting wood. We used all kinds of saws, found out how they interacted with different kinds of wood, and learned to cut accurately and smoothly. I got pretty good at cutting wood. The next four weeks we learned to plane wood. We used all kinds of planes, on many different kinds of wood. I got pretty good at planing wood. 'Joints' was a difficult course. It took eight weeks, and we learned many kinds of joints. I was quite good at making joints. We did courses on other things too: sanding, turning, polishing, gluing and so on. Finally, we had an examination. We had to use all these skills. I did reasonably well, and came fifth in the class.

After the course ended, I went to see the Director. I told him that I quite liked the course in a way, though some of the students were turned off by it all. But really, I said, I took the course because I wanted to make a table. He said that the top two or three went on to do things like that. I began to get mad. I said: "What did we learn all that stuff for?". He said: "Our course prepares people to make tables." His face got larger and larger. He began to fill the room. I got scared. Then I woke up.

This was worrying. I discussed it with my colleagues. A psychiatrist took me back to my childhood. But no-one could explain why a professor of mathematics should have a nightmare like that.

Problems Drive Solutions 1997

Paul Bolchover and Sean Blanchflower

1 The completed phrases are as follows:

a 1729 = Number of Hardy's Taxi
b 4.6692... = Rate of Period Doubling in some Dynamical Systems
c 0.00000007151 = Probability of Winning the National Lottery
d $2^{3021376}(2^{3021377}-1)$ = Largest Known Perfect Number
e 3.14159... = Ludolphine Number
f 13 = Archimedean Polyhedra
g $(1+\sqrt{5})/4$ = Cos of Pi By Five
h $\sqrt{2}/12$ = Volume of Regular Tetrahedron of Side One
i 23 = Hilbert Problems
j 13 = Books of Euclid's Elements
k 6 = Regular Solids in Four Dimmensions
l 25 = Primes Below One Hundred

2 The pairing between equations and locii is as follows:

a	i	**i**	xiv
b	xiii	**j**	iii
c	viii	**k**	v
d	xv	**l**	xi
e	x	**m**	ii
f	xvi	**n**	iv
g	vii	**o**	xi
h	ix	**p**	vi

3 The missing terms in the sequences are shown below in boldface, with an explanation of the sequence in parentheses:

a 100, 200, 300, 301, 302, 303, 304, 309, **350**, **351** (roman numbers in alphabetical order)
b 5, 6, 5, 6, 5, 5, 7, **6**, **6** (number of letters in the ordinals)
c 1, 3, 4, 9, 10, 12, 13, 27, 28, **30**, **31** (binary integers treated as base 3, converted to decimal)
d 0, 2, 3, 4, 5, 5, 7, 6, 6, 7, 11, **7**, **13** (sum of factors in prime factorisation)
e 1, 6, 15, 28, 45, **66**, **91** (alternate triangular numbers $n(2n-1)$)
f 1, 2, 5/2, 17/6, 91/30, **379/120**, **5047/1560** (sums of 1/Fibonacci numbers so the nth term is $1/\sum_1^n 1/F_i$)

4 The three possibilities for the number represented by the letters ABCD are 2025,3025, and 9801.

5 The results of the four real ludo matches are shown below:

Match 1:	1: Alice	Match 3:	1: Alice
	2: Harriet		2: Ben
	3: Ben		3: Frances
	4: Edward		4: George
Match 2:	1: George	Match 4:	1: Harriet
	2: David		2: Edward
	3: Frances		3: David
	4: Carol		4: Carol

6 The minimum and maximum values of the sum of the numbers represented by the letters (FOURTEEN+NINETEEN) are 116971336 and 128164326 respectively.

7 The area of ABCD is 1350.

8 The radius of the largest sphere which can be placed in the gap between the other spheres is $(\sqrt{6}-1)/4$.

9 Both the exact value and the evaluation to ten decimal places were required in the solution:

$$\int_0^1 \frac{2x^9(1-x^2)^4}{1+x^4}\,dx = \frac{22}{7} - \pi = 0.0012644893 \text{ (to 10 d.p.)}$$

10 The completed magic square, which contains all the required diagonals is:

19	06	15	22	31	18
34	01	33	32	02	09
14	10	26	11	27	23
16	30	20	17	07	21
03	35	04	05	36	28
25	29	13	24	08	12

11 The completed cross-number is as follows:

1. 7	2	2. 9		3. 3	2	4. 4
8		5. 6	7	6		8
6. 4	4	1		7. 1	4	4

12 The largest number for which every two consecutive digits form distinct two-digit primes is 619737131179.