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Editorial

The very first issue of our annual(-ish) journal (published in 1939, price 6*d.*) states the task of *Eureka* and its Editors as "to [be] interesting to every Cambridge mathematician... to link together students, researchers and dons, other English and foreign universities... [and] to stimulate informed discussion, especially as to Cambridge questions". Looking at the varied backgrounds of contributors to this issue, I think that the second aim is here admirably fulfilled. But are the first and third equally so? I should hope that the "mathematician" of the first aim refers not only to those who follow the Mathematical Tripos but also to those who study mathematics either as part of another course or in their own time. In this respect then, perhaps the current *Eureka* is a little lacking. Can I therefore exhort the readership of this issue to encourage friends who may not necessarily even be members of the Archimedeans to read and possibly to contribute in some guise to this journal? The third aim is perhaps more vague. This issue of *Eureka* contains articles mostly of a mathematical nature, but in the past we have seen mathematically-inspired fiction, poetry and other artworks, opinion pieces on Cambridge teaching and the examination system and other less formal outpourings alongside the meatier mathematical articles. These "stimulate informed discussion" just as effectively as the many wonderful expositions of mathematical snippets in these pages, and can perhaps help to extend the definition of the "mathematician" to whom *Eureka* aspires to be interesting.

In conclusion – please write something! Write anything! The diverse talent of mathematicians is nowhere more evident than in Cambridge; I hope that *Eureka* can continue to draw on (and draw out) that talent in fulfilling its aims for many years to come.

Cambridge
April 2005

Acknowledgements
My job as Editor has been made immeasurably easier by the willingness of people to contribute to the journal, for which I must thank all whose names appear in the following pages; their patience and promptness in replying to my own often overdue emails have been much appreciated. Next I should like to thank the invisible army of proof-readers, in particular Joseph Myers, Jordan Skittrall, my predecessor Vicky Neale and the current Assistant Editor David Chow, whose work in checking several versions of the text has removed many errors, typos and obscurities (however, any that remain are of course entirely my responsibility). I should reiterate my thanks to Vicky for much useful advice and LaTeX source for some of the pages, and to David for excellent advice on issues of typesetting, clarity and style throughout the year, as well as for no fewer than four submissions to this issue! Last-minute thanks go to Alex Shannon for coming up with an excellent cover design when I had just given up trying to draw stick people with fractal shoes. All of the Committee have been very supportive of the publication of *Eureka* 57; I hope that it lives up to their expectations.

The Archimedeans 2004–2005

Jenny Gardner (Secretary 2004–2005)

The Archimedeans celebrated the end of exams last June with an evening punt trip to Grantchester, including a pub meal when we got there, and a pirate flag to compete over along the way. The following week our garden party was held, where much hilarity was had with Twister and Jenga, although sadly we were unable to obtain a giant Jenga set.

The Michaelmas Term saw many new faces join the Society, as well as a series of high profile speakers. Talks by Marcus du Sautoy, Brian Bowditch, Simon Singh and John Conway were all very well attended. The Puzzle Hunt at the end of term yet again produced several amusing mathematical stories, and was followed by a Christmas Party complete with mulled wine and mince pies. QARCH was also published, for the first time in many years.

The Triennial Dinner, held at Emmanuel College, provided good food, drink and amusement for Society Members in February. We were entertained by a speech by Imre Leader, as well as a "Geometrical Fugue", performed by four Committee Members. The revelry continued well into the night, and provided a good source of gossip for several days afterwards. A few weeks later we were joined by some of the Invariants, our Oxford counterparts, for a pub lunch, followed by a very popular Problems Drive. Having once again made contact with the Invariants, we hope to hold more joint events with them in the future, and in particular we are planning to challenge them to a croquet match this Easter Term.

Henrik Jensen, Helen Byrne and Ian Grant all gave entertaining talks to the Society in the Lent Term. On behalf of the Society, I'd like to thank those who have spoken to the Archimedeans this year. A new subgroup, Seminars for Undergraduates, was also born, and its first talk was given by Aaron Lauda. The subgroup aims to provide discussion-based seminars to give undergraduates an insight into current mathematical research. The first seminar was very well received, and more are planned for the future. The Puzzles and Games Ring has also continued to meet, and there have been regular Bookshop sales throughout the year.

Overall, the Society has expanded throughout the year, with old traditions being continued or restarted as well as new initiatives made. Next year promises to be another exciting year, and I am confident that the Archimedeans will continue to grow and flourish.

Erratum to Eureka 56

On page 18, the condition on U_z in the proof of Lemma 4 should be $f(z) \leq x$, not $z \leq f(x)$.

Thanks to Demetres Christofides for pointing this out.

Fractals, Image Compression and the Contraction Mapping Theorem

Alexander Shannon

One of the most often cited applications of the study of fractals is that of their use in image compression. Such an application is not surprising, since seemingly complicated and intricate fractal images have relatively simple mathematical descriptions in terms of iterated mappings. Given also that fractals have been found to model well a wide variety of natural forms (a famous example being that of Figure 1), it seems natural that we should try to exploit their self-similar properties to encode images of such forms.

Figure 1: A fractal fern

We examine a simple example of a fractal, the Koch curve, to illustrate the principle of encoding a fractal image. Referring to Figure 2, we construct the Koch curve by first taking a line segment of length 1, K_0. We then construct K_1 by combining the images of this line segment under four transformations, each involving a dilation of factor $\frac{1}{3}$ composed with either or both of a rotation and translation. Combining the images of K_1 under the same four transformations yields K_2, and the Koch curve itself (K_∞) is the limit of this process as it is iterated. (Peitgen, in [4], calls this method of drawing fractals the 'Multiple Reduction Copy Machine' or MRCM.) We can see that this object is self-similar, in the sense that we can find arbitrarily small portions of the curve that are related to the whole by a similarity transformation. The fractal fern of Figure 1 is also the limit of four affine transformations iterated in the same manner. Since each affine transformation may be represented by a 2×2 matrix giving the homogeneous part of the transformation and a 2-component vector giving the inhomogeneous (translation) part of the transformation, a figure that is the limit of n iterated affine transformations can be encoded as a collection of $6n$ real numbers – a much more efficient encoding than a pixel-by-pixel representation. These ideas also generalise in an obvious manner to subsets of higher dimensional Euclidean space.

We might then ask whether we can measure how 'close' a perfectly self-similar or self-affine fractal image is to a given 'imperfect' real-life image that we are trying to approximate. We might also ask how many iterations of the kind described above we need to carry out to get a reasonable approximation of the limiting set. More theoretically, we might question whether we can be sure that such iterations will indeed tend to a definite limit, and, given that any such limit will be invariant under the iteration, whether it matters with what set we start. Could we, for example, have begun our construction of the Koch curve with a circle rather

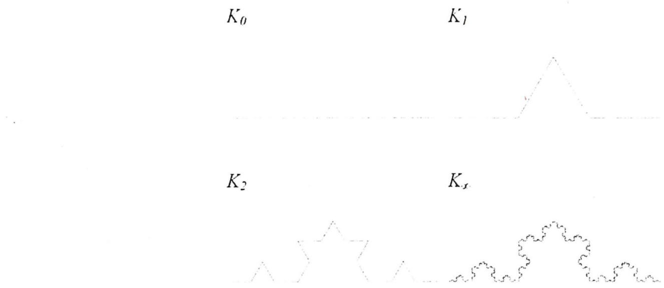

Figure 2: Construction of the Koch curve

than a line segment? In this article, we shall see that, by considering subsets of Euclidean space as points in a metric space, we can measure how different two images are, and by applying the contraction mapping theorem, we can see that limit sets of the sort described above do exist, that our starting point in their construction does not matter, and we can also obtain an estimate for how rapid the convergence is.

1 Definitions

For reference, we enumerate here a few standard definitions and theorems that we shall use later.

Definition 1 *A metric space is an ordered pair* $(\mathfrak{X}, d)$*, where* $\mathfrak{X}$ *is a set and* $d : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathbb{R}$ *is a function with the following properties:*

(i) $d(x, y) \geq 0 \quad \forall x,\ y \in \mathfrak{X}$ *with* $d(x, y) = 0$ *if and only if* $x = y$*;*
(ii) $d(x, y) = d(y, x) \quad \forall x,\ y \in \mathfrak{X}$*;*
(iii) $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x,\ y,\ z \in \mathfrak{X}$.

The notion of convergence of a sequence to a limit carries over to metric spaces in an obvious way, as does the following related notion:

Definition 2 *Let* (x_n) *be a sequence of points in a metric space* $(\mathfrak{X}, d)$*. We say that* (x_n) *is Cauchy if, given* $\epsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that for all* $m,\ n \geq N$*,* $d(x_m, x_n) < \epsilon$.

Clearly every convergent sequence is a Cauchy sequence. The converse is also true for an important class of metric spaces:

Definition 3 *We say that a metric space* $(\mathfrak{X}, d)$ *is complete if every Cauchy sequence in* $\mathfrak{X}$ *converges.*

We remark that the metric space formed by $\mathbb{R}^n$ with the usual Euclidean metric is complete.

Definition 4 *Let* $(\mathfrak{X}, d)$ *be a metric space. Then* $f : \mathfrak{X} \longrightarrow \mathfrak{X}$ *is a contraction if there exists a non-negative real number* $c < 1$ *such that* $d(f(x), f(y)) \leq c \cdot d(x, y)$ *for all* $x,\ y \in \mathfrak{X}$.

Our central theorem tells us something about the behaviour of contractions under iteration (for a proof, see, for example, [3]).

Theorem 5 (The Contraction Mapping Theorem) *Let $(\mathfrak{X}, d)$ be a non-empty complete metric space and $f : \mathfrak{X} \to \mathfrak{X}$ a contraction. Then there exists a unique $x_0 \in \mathfrak{X}$ such that $f(x_0) = x_0$, and furthermore, $\lim_{n\to\infty} f^n(x) = x_0$ for all $x \in \mathfrak{X}$.*

This has a corollary to which we will refer in the final section:

Corollary 6 *Let $(\mathfrak{X}, d)$ be a non-empty complete metric space and $f : \mathfrak{X} \to \mathfrak{X}$ such that f^n is a contraction. Then the same conclusions hold as for Theorem 5.*

For most of the time, we shall restrict our attention to *compact* subsets of metric spaces.

Definition 7 *Let $(\mathfrak{X}, d)$ be a metric space. Then we say $A \subseteq \mathfrak{X}$ is compact if every covering of A by open sets has a finite subcovering.*

The important properties of compact sets which we need are that they are closed and bounded.

2 Hausdorff Distance

Our starting point is a way of turning a collection of subsets of Euclidean space into a complete metric space, so that we can talk about limits and convergence, and make use of the considerable information provided by the contraction mapping theorem. The concept we require is due to Hausdorff, who formulated a notion of 'distance' between compact subsets of a metric space which makes the set of compact subsets of a given metric space into a metric space itself. Furthermore, if our initial metric space is complete, then so is the space of compact subsets with the Hausdorff metric.

We require a further concept before introducing the definition of Hausdorff distance itself.

Definition 8 *Let A be a subset of a metric space $(\mathfrak{X}, d)$. The ϵ-collar of A, denoted A_ϵ, is the set $\{x \in \mathfrak{X} : \exists a \in A \text{ with } d(a,x) \leq \epsilon\}$, i.e., the set of all points at a distance at most ϵ from the set A. (See Figure 3)*

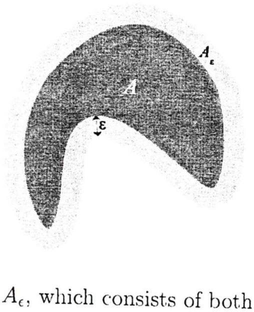

Figure 3: The ϵ-collar of a set A, A_ϵ, which consists of both the light and dark shaded areas.

Definition 9 *Let A and B be compact subsets of a metric space $(\mathfrak{X}, d)$. If we write $\rho'(A, B) = \inf\{\epsilon > 0 : A \subseteq B_\epsilon\}$ then the Hausdorff distance between A and B, $\rho(A, B)$, is defined by $\rho(A, B) = \max\{\rho'(A, B), \rho'(B, A)\}$.*

It follows straightforwardly from the definition that ρ' satisfies all the axioms for a metric in Definition 1 except for *(ii)* (the details are given explicitly in [2] and in [3] as an exercise), so the final part of the definition is essentially a symmetrisation. An alternative definition sometimes used (for example in [3]) but which does the same job is $\rho(A, B) = \rho'(A, B) + \rho'(B, A)$. The proof that the resulting metric space inherits completeness is again given in [2] and as an exercise in [3].

3 The Hutchinson Operator

Now that we have a way of measuring 'closeness' of compact subsets of metric spaces, our next task is to show that the iterated transformation applied in Figure 2 to construct the Koch curve is indeed a contraction, so that we may apply Theorem 5. The following treatment follows quite closely that of [4]. We work in $\mathbb{R}^m$.

We have a collection of affine transformations, $T_1, T_2, \ldots, T_n$, and at each iteration we apply the transformation

$$T : A \mapsto \bigcup_{i=1}^{n} T_i A.$$

(This is known as the Hutchinson operator, after Hutchinson, who first analysed its properties.) We impose the condition that each T_i should itself be a contraction with respect to the Euclidean metric, with constant $c_i < 1$.

We now show that T is a contraction with constant $c = \max\{c_1, c_2, \ldots, c_n\}$ on the metric space of compact subsets of $\mathbb{R}^m$ equipped with the Hausdorff metric. (Compare Figure 4 for the following.) Let A, B be compact subsets of $\mathbb{R}^m$ with $\rho'(B, A) = \delta$. Then for any $\epsilon > \delta$ we have $B \subseteq A_\epsilon$. Clearly then $T_i B \subseteq T_i A_\epsilon$ for each i, but since T_i is contractive on $\mathbb{R}^m$, $T_i A_\epsilon \subseteq (T_i A)_{\epsilon_i}$, where $\epsilon_i = c_i \epsilon \leq c\epsilon$. Hence $T_i B \subseteq (T_i A)_{\epsilon_i} \subseteq (T_i A)_{c\epsilon}$, yielding

$$\bigcup_{i=1}^{n} T_i B \subseteq \bigcup_{i=1}^{n} (T_i A)_{c\epsilon} = \left(\bigcup_{i=1}^{n} T_i A \right)_{c\epsilon}.$$

So $TB \subseteq (TA)_{c\epsilon}$ for all $\epsilon > \delta$, and hence $\rho'(TB, TA) \leq c\delta$. Therefore $\rho(TA, TB) \leq c \cdot \rho(A, B)$ and so T is indeed a contraction.

An important practical observation which can be made from the above proof is that the contraction constant calculated for T is equal to the largest of the individual contraction constants of the transformations T_i. It is clear from the form of the proof that, in general, we can do no better than this. In the usual proof of the contraction mapping theorem, it is shown that, for a contraction f with constant c,

$$d(f^n(x), f^{n+k}(x)) \leq d(x, f(x)) \frac{c^n}{1 - c}.$$

Since this inequality holds for all k, the expression $c^n/(1 - c)$ provides an estimate for how quickly the iterations converge to the unique fixed point. As might be expected, we see that the larger the constant c, the slower the convergence. Hence the MRCM method of drawing fractals is only as rapid as is allowed by the 'least contractive' contraction. It is, however,

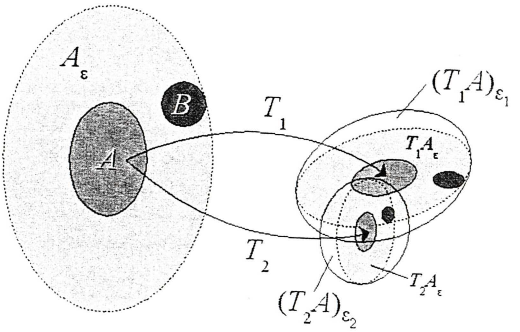

Figure 4: The Hutchinson Operator as a Contraction

worth remarking that a given transformation may or may not be contractive, depending on the choice of metric, and that the contraction constants will vary according to the metric used. Since the notion of Hausdorff distance works for any metric space, not just $\mathbb{R}^m$ with the Euclidean metric, we may certainly replace the Euclidean metric in the above analysis with any other making $\mathbb{R}^m$ into a complete metric space, to be able to draw conclusions about the convergence properties of a wider variety of Hutchinson operators.

4 Julia Sets

We conclude with some brief, informal remarks about how the above ideas may be applied to producing images of another rather famous class of fractals. For a given polynomial $f : \mathbb{C} \to \mathbb{C}$, the Julia set of f, $J(f)$, is the closure of the set of repelling (unstable) fixed and periodic points of f. This is non-trivially equivalent to the definition as the boundary of the basins of attraction of the attractive fixed points of f (for details see [1]), and the set $J(f)$ has the property that $f(J) = f^{-1}(J) = J$. The most famous examples of these objects are those associated with the mapping $f : z \mapsto z^2 + c$ for various $c \in \mathbb{C}$ (a particular example is shown in Figure 5). In this case we notice that the inverse mapping, $f^{-1} : z \mapsto \{\pm\sqrt{z - c}\}$ seems to play the rôle of a non-linear Hutchinson operator, in that each point (other than c itself) has two images, and the fractal of interest is invariant under the transformation.

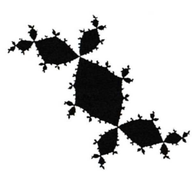

Figure 5: A Julia Set

We might well then ask whether the mapping is contractive. Here a partial answer is suggested by the theory of conformal mappings, which tells us that for a conformal mapping $g : \mathbb{C} \to \mathbb{C}$, the approximate scaling in length near a point $z_0 \in \mathbb{C}$ is $|g'(z_0)|$. The criterion for a fixed point z_0 of a mapping g to be attractive, viz. $|g'(z_0)| < 1$, is therefore the same as the criterion for the mapping to be locally contractive. Any point close to $J(f)$ is, by definition, close to some repelling periodic point of f (whose period we shall denote by p), which in turn will be an attractive periodic point of $f_1^{-1} : z \mapsto +\sqrt{z-c}$ and $f_2^{-1} : z \mapsto -\sqrt{z-c}$. Hence the iterate T^p of the Hutchinson operator T defined by these two mappings will be a local contraction, and so Corollary 6 suggests that, at least if we consider sets not 'too far' in terms of Hausdorff distance from $J(f)$, the iterations will converge in the same manner as for the self-affine fractals discussed above. In fact the convergence is very good, and although after a finite time the iterates do not in general approximate all parts of the Julia set evenly, this is how many fractal drawing packages produce their images of Julia sets.

References

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[2] F. Hausdorff, *Set Theory*, Chelsea Pub. Co. (1962).

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[4] H.-O. Peitgen, H. Jürgens and D. Saupe, *Chaos and Fractals*, Springer-Verlag (1992).

Seminars for Undergraduates

At meetings of this newly-formed subgroup of the Archimedeans, young researchers in mathematics offer exciting seminars on their areas of work and interest. Last term Aaron Lauda gave the highly successful inaugural talk on the role of category theory in modern physics ("Frobenius algebras and topological quantum field theory"). This term will see contributions from speakers David Kagan and Konstantin Ardakov. Accessible to an undergraduate audience, these seminars are perfect for anyone considering research as a possible career. Information can be found on the website (see inside front cover).

If you are a postdoctoral researcher or PhD student and would like to give a talk to the group, please contact the organisers (`archim-seminars@srcf.ucam.org`).

Birthdays on Jupiter

Ian Stewart

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Jupiter's 'year', the time it takes to orbit the Sun once, contains approximately 10477 Jovian 'days', the time it takes for the planet to spin once on its axis. The bloons – creatures whose cells are filled with hydrogen and so float in Jupiter's predominantly hydrogen-helium atmosphere – therefore have 10477 different birthdays. And this has led to the bloon mathematicians becoming extraordinarily rich. Whenever a bloon mathematician is at a party with more than 121 bloons present, he/she/it places bets with all and sundry that at least two of the creatures present have the same birthday (not counting the year). Because 121 is so small compared to 10477, nobody who is not in on the secret of this particular scam believes that the odds actually favour the mathematician. Despite several Jovian centuries of this con-trick, the punters continue to fall for it, because they never paid attention in their school probability theory lessons and they think that guesswork is as good as anything else – after all, every bloon is entitled to his/her/its own opinion, isn't he/she/it?

The mathematicians smugle inwardly – a smugle is the bloon equivalent of a smug smile – knowing that entitlement to an opinion is no guarantee that it's correct. In fact, they smugle all the way to the bank.

Ths trick has not gone unremarked by a species of alien invader from the planet Neeblebruct, whose invisible fleet has been circling Jupiter for half a Jovian century. Over the years they have abducted many fortytwos of Jovian mathematicians in the hope of discovering their secret. The snag is that a Neeblebructian year contains exactly $42^4 = 3111696$ Neeblebructian days, and nobody has managed to work out what the correct substitute for 121 is.

Until now.

You've almost certainly met the same problem with the Earth year of 365 days (not counting leap years). There, the critical number of people is 23: see below for a proof. Again, the number required is remarkably small. How small? Let's find out.

In order to win the bet, in the long run, the mathematician of the appropriate species must arrange for the probability of two or more identical birthdays to be greater than 1/2. The 'break even point' occurs when that probability equals 1/2. So we are led to formulate:

Problem 1 *With n possible birthdays, how many entities must there be in a room in order for the probability that two or more have the same birthday to be better than 50%?*

The usual idealization of this question assumes that that any birth date has the same probability, and I'm going to stick to that convention here. (If not, the chance of identical birthdays increases, so we're not giving much away.) It's also standard to ignore the effect of leap years, and I'll make that simplification too. Finally, we assume that all individuals have statistically independent birthdays. (That is not the case on, for instance, the iceworld of Gnux Prime, where each new generation emerges simultaneously from its underground hibernation-tube and distinct generations don't go to the same parties – rather like a cross between periodical cicadas and humans on Earth. As soon as two Gnuxoids enter a party-space, the probability of them sharing the same birthday becomes 1. So Gnuxoid mathematicians would always win the bet, but every new generation of punters quickly gets wise to this scam.)

Given all this, the trick for solving Problem 1 is to imagine the entities entering the party-habitat one at a time, and to work out – at each stage – the probability that all birthdays so far are *different*. Subtract the result from 1 and you get the probability that at least two are equal.

Let's do it with humans, following Mosteller [2] Problem 31.

When the first person enters the party, the probability that their birthday is different from that of anyone else present is $\frac{365}{365} = 1$. Because nobody else *is* present, right?

When the second person enters, their birthday has to be different, so there are 364 choices out of 365 and the probability is therefore $\frac{365}{365} \cdot \frac{364}{365}$.

When the third person enters, they have only 363 choices, and the probability of no duplication so far is $\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}$.

The pattern should now be clear. After k people have entered, the probability that all k birthdays are distinct is

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-k+1}{365}.$$

Writing $n = 365$ this becomes

$$p(k) = \frac{n(n-1)\cdots(n-k+1)}{n^k},$$

and we want the first k for which $p(k) < 1/2$.

Clearly $p(k)$ decreases as k increases. Direct calculation shows that $p(22) = 0.524305$ and $p(23) = 0.492703$. So the required number of people is 23.

This number seems surprisingly small to most sentient entities who meet this problem for the first time. Mosteller [2] Problem 32 suggests that this may be because the problem gets confused with a different one: how many people do you have to ask for the probability that one of them has the same birthday *as you* to be better than 50%? We'll come back to that shortly: see Problem 3 below.

A common variant of Problem 1 is:

Problem 2 *With n possible birthdays, what is the* expected *number of people in a room such that at least two share the same birthday?*

This time the answer for $n = 365$, with the same idealizations, is 23.9: see Problem 101 of Newman [3]. This is close enough to 23 that the two questions sometimes get confused, but in general their answers may differ.

On Mars, whose 'year' contains 670 days, the solution to Problem 1 is 31. On Jupiter, whose year contains 10477 days, it is 121. And on a planet whose year contains n days... what is the correct number? If we solve this and set $n = 3111696$, then we'll keep the Neeblebructians happy.

We can't expect an exact formula, but we ought to be able to find a good approximation. Newman [3] shows that for Problem 2, the expected number k of samples is asymptotic (for large n) to

$$\sqrt{\frac{\pi}{2}}\sqrt{n},$$

by using Euler's integral for the gamma function and the dominated convergence theorem. We'll derive a similar estimate for Problem 1, which will also show that the two problems generally have different answers.

For the same reasons that apply on Earth, but with 365 replaced by n, we want

$$p(k) = \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k} \sim \frac{1}{2},$$

which we rewrite as

$$\log n + \log(n-1) + \log(n-2) + \cdots \log(n-k+1) \sim k\log n - \log 2.$$

Approximating the sum by an integral leads to:

$$\begin{aligned} \int_{n-k}^{n} \log x dx &\sim k\log n - \log 2 \\ [x\log x - x]_{n-k}^{n} &\sim k\log n - \log 2 \\ n\log n - (n-k)\log(n-k) - k &\sim k\log n - \log 2. \end{aligned}$$

Therefore

$$-(n-k)\log\left(1-\frac{k}{n}\right) \sim k - \log 2.$$

Expanding in a power series to order 2 in k we get

$$(n-k)\left(\frac{k}{n}+\frac{k^2}{2n^2}\right) \sim k - \log 2,$$

so that, discarding the cubic term in k,

$$\frac{k^2}{2n} \sim \log 2,$$

leading to the asymptotic formula

$$k \sim \sqrt{\log 4}\sqrt{n}.$$

To compare Problems 1 and 2, observe that

$$\begin{aligned} \sqrt{\frac{\pi}{2}} &\approx 1.2533141373 \\ \text{and } \sqrt{\log 4} &\approx 1.1774100225, \end{aligned}$$

which differ by about 6%.

The asymptotic formula for Problem 1 yields the following results:

Earth: When $n = 365$ we get $k \sim 22.4944$.

Mars: When $n = 670$ we get $k \sim 30.4765$.

Jupiter: When $n = 10477$ we get $k \sim 120.516$.

Recall that the *ceiling function* $\lceil x \rceil$ is the smallest integer $\geq x$. In all three cases, the ceiling function $\lceil \sqrt{\log 4}\sqrt{n} \rceil$ is *exact.*

Could this provide the exact formula which I said shouldn't be expected?

Unfortunately not.

Experiment reveals that this expression is too small (by 1) when $n = 24, 25$ and $253 \leq n \leq 259$, for example. To find out why, we need to analyze the error in the approximation, but I won't attempt that here. However, I can now answer the problem that has baffled

Neeblebructian science: the number of entities needs to be $\sqrt{\log 4}\sqrt{3111696} = 2076.95$, whose ceiling is 2077. Since we've not estimated the error, it's worth checking more closely: in fact,

$$p(2076) = 0.500407, \qquad p(2077) = 0.500074, \qquad p(2078) = 0.49974,$$

so the ceiling estimate is too small by 1.

This error arises because 2076.95 is only slightly less than the integer 2077. In fact, further experiment shows that $\lceil\sqrt{\log 4}\sqrt{n}\,\rceil$ is slightly too small whenever $\sqrt{\log 4}\sqrt{n}$ gets close to, but is smaller than, an integer. To study this situation, we need a more accurate version of the calculation, using the Euler-Maclaurin summation formula (see Graham *et al.* [1]). But I'm going to leave that analysis to you, mainly because I haven't done it myself yet.

Instead, let's relate everything done so far to the version of the problem that Mosteller believes is often confused with Problem 1:

Problem 3 *How many entities, other than you, must there be in a room in order for the probability that at least one of them has the same birthday as you to be better than 50%?*

The chance that any given person does not have the same birthday as you is $\frac{364}{365}$. So the required number K should satisfy

$$\left(\frac{364}{365}\right)^K \sim \frac{1}{2},$$

which leads to $K = 253$. If you are a Martian, the answer is 465. If you're a Jovian it is 7262. And if you have the privilege to be a Neeblebructian, it is 2156863.

As it happens, K is the 22nd triangular number (that is, it is equal to the sum $1+2+3+\cdots+22$). So the answer to Problem 3, for Planet Earth, is the $(-1+$Answer to Problem 1)th triangular number. In [4] I unwisely remarked that this must be a coincidence. Joe Gerver pointed out to me that it's not. Here's why.

With n days in the year, we have

$$\left(\frac{n-1}{n}\right)^K \sim \frac{1}{2}.$$

We could take logs and solve this for K. Or, supposing that n is large, we can use the approximation

$$\left(1-\frac{1}{n}\right)^n \sim \mathrm{e}^{-1}.$$

So the above equation becomes

$$\mathrm{e}^{-K/n} \sim \frac{1}{2}$$

or

$$K \sim n \log 2.$$

Now the kth triangular number is

$$\frac{k(k-1)}{2} \sim \frac{k^2}{2} \sim \frac{n \log 4}{2} = n \log 2 \sim K,$$

and the 'coincidence' is explained. A more careful analysis, which I leave to you, shows that the approximation is better than you might expect when K happens to be a triangular number. So that's the *real* coincidence here.

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Paul's Letter to the Romans, Chapter 5 Verse 8

(The New Mathmo Translation)

Stephen Burgess

$$\begin{gathered}
\exists L_G : B(0, r_e) \to \mathbb{R} \\
\text{s.t. } \forall x \in B(0, r_e),\ \|x - G\| > N,\ \forall N \in \mathbb{R} \\
\text{but given } \epsilon > 0,\ \|L_G(x) - G\| < \epsilon, \\
\text{where } x_J \in \bar{B}(0, r_e) \setminus B(0, r_e), \\
x_J \leftrightarrow G, \\
\text{and } \ker(L_G) = \{x_J\}.
\end{gathered}$$

Read: "There exists a function Love of God acting from the open ball of the earth which is real-valued such that for all members of the earth, despite being arbitrarily far away from God, the application of the Love of God function brings them arbitrarily close to God, with one special point, denoted J, who was in the world but not of the world and who corresponds to God, who was the only thing which, when applied by the Love of God, was reduced to nothing."

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Problems Drive 2004

David Chow and Vicki Wright

Question 1. Early calculation?

MS UR TF RD GU RE DU XC EV AM TL SB NE JT SN XJ RC NE WT NW RO WT O?

Question 2. Consider the function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x^x$. Find the value of its tenth derivative at $x = 1$.

Question 3. Match each of the following quotes with the person from which it came.

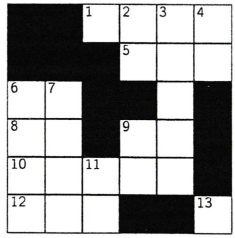
(a) Perfect numbers like perfect men are very rare.
(b) God made the integers; all else is the work of man.
(c) If your experiment needs statistics, you ought to have done a better experiment.
(d) A mathematician is a machine for turning coffee into theorems.
(e) No, it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.
(f) The theoretical broadening which comes from having many humanities subjects on the campus is offset by the general dopiness of the people who study these things.
(g) From the intrinsic evidence of his creation, the Great Architect of the Universe now begins to appear as a pure mathematician.
(h) I have found it!
(i) A theoretical physicist is someone who has no respect for mathematics and can't do experiments.
(j) Do not worry about your problems with mathematics, I assure you mine are far greater.
(k) If I have seen further than others, it is by standing on the shoulders of giants.
(l) I have found a marvellous proof of this which this margin is too narrow to contain.

(i) Archimedes.
(ii) René Descartes.
(iii) Albert Einstein.
(iv) Paul Erdős.
(v) Pierre de Fermat.
(vi) Richard Feynman.
(vii) James Jeans.
(viii) Leopold Kronecker.
(ix) Isaac Newton.
(x) Srinivasa Ramanujan.
(xi) Ernest Rutherford.
(xii) Neil Turok.

Question 4. Find the value of

$$\sum_{n=1}^{\infty} 4^n \sin^4\left(\frac{\theta}{2^n}\right).$$

Question 5. Using the clues below, complete the following grid:



ACROSS

(1) Product of the first x digits of π, divided by x^2, where $x =$(8 across), after rounding π to x significant figures.
(5) Largest prime less than (8 across)^3.
(6) (2 down)+(9 down).
(8) A useful number.
(9) A square number.
(10) Last 5 digits of (1 across)^3.
(12) $n^3 + n^2$ for some $n \in \mathbb{N}$.

DOWN

(2) One less than (11 down).
(3) First rotationally symmetric number greater than (1 across)$\times$((4 down)+(8 across)).
(4) A clueless number.
(6) A Fibonacci number.
(7) Sum of the cubes of the first (8 across) natural numbers including 0.
(9) A prime number.
(11) (13 other)$\times$(8 across).

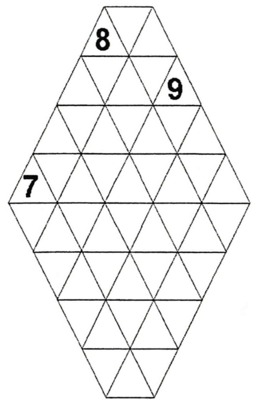
OTHER

(13) Lowest common divisor of (1 across) and (7 down).

Question 6. Find one more term in each of the following sequences:

(a) 2, 3, 5, 7, 1, 1, 1, 3, 1, 7, 1, 9, ...
(b) 510, 152, 025, 303, 540, ...
(c) 6, 90, 945, ...
(d) 1, 1, 1, 2, 1, 1, 1, 3, 2, 1, ...
(e) -4, 7, 16, 118, 127, 208, 226, 307, ...

Question 7. Enter single digit numbers to complete the grid making the figures in each of the 16 hexagons add up to 29. No two numbers in one hexagon may be the same and 0 may not be used.



Question 8. There are 5 students: Adam, Bob, Charles, David and Edward. Each student owns a different copy of *Eureka* with an edition number between 48 and 52. The 5 students attend 5 colleges, Christ's, St John's, King's, Sidney Sussex and Trinity, and each is keen on one mathematician from the following list: Euler, Fermat, Galois, Hilbert and Lagrange.

Using the 7 clues below, find the name, college and edition of *Eureka* owned by the student who likes Hilbert.

(1) The student from St John's, whose role model is not Hilbert, is the proud owner of a copy of *Eureka* 51.
(2) The Trinitarian does not possess a version of the oldest *Eureka* in the group.
(3) David, a Fermat enthusiast, is neither from Sidney Sussex, nor does he own *Eureka* 49.
(4) The sale of *Eureka* 50 involved a purchaser from King's who does not like Fermat and whose name, not alphabetically first, is in an odd position.
(5) Charles, a Johnian, likes neither Lagrange nor Euler, and the student from Christ's does not own a copy of the newest edition of *Eureka*.
(6) The student from the, alphabetically speaking, last college, proud owner of *Eureka* 49, is not Adam, who owns a copy of *Eureka* which, when the two digits of the edition number are summed, does not give an even result.
(7) Bob likes neither the first nor the last, alphabetically speaking, mathematician and possesses neither a copy of the oldest nor the newest edition of *Eureka*.

Question 9. A, B, C, D and E each represent distinct digits. If $A = 1$ and $B = 2$, then $AB = 12$, etc. Excluding the possibility of leading zeros, find their values given that

$$(AB + D)^E = ABCDE.$$

Question 10. Consider the equation

$$\exp(\pi z/2) = z.$$

This has the obvious solutions $z = \pm i$. Find some more solutions, rounding them to the nearest Gaussian integer $x + iy$ with $x, y \in \mathbb{Z}$.

Question 11. Consider $\mathcal{P}_2[-1, 1]$, the set of all polynomials in a real variable x which have real coefficients, are of degree at most 2 and have domain $[-1, 1]$. Turn this into a real vector space with the natural definitions of addition and scalar multiplication. Define the inner product $\langle p, q\rangle$ of two vectors $p, q \in \mathcal{P}_2[-1, 1]$ by

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\mathrm{d}x.$$

We say that p and q are orthogonal if $\langle p, q\rangle = 0$.

Malcolm is asked to find an orthogonal basis for $\mathcal{P}_2[-1, 1]$. Misunderstanding the question, he comes up with a basis for $\mathcal{P}_2[-1, 1]$ such that when the graphs of the basis polynomials are plotted to scale, they pairwise intersect each other at right angles. By chance this turns out to be an orthogonal basis as well! Find such a basis.

Question 12. Remarkably, it's sometimes possible to come across mathematics which makes sense when read upside down! An example is the statement $d + p = p + d$, which, when read upside down, becomes $p + d = d + p$. However, both statements are trivial and essentially say the same thing.

Find an example of a mathematical statement such that:

(a) It makes sense as a mathematical statement when viewed upside down.
(b) Both statements are non-trivial.
(c) Both statements are essentially different, for example not simply rearrangements or relabellings of each other.

The Mathematics of Juggling

Owen Jones

There are many ways in which maths can be applied to juggling. This article is predominantly about **siteswap notation**, a simple way of writing down juggling patterns that has improved communication between jugglers and allowed many new patterns to be discovered. To simplify matters we shall first consider only patterns in which one ball is thrown on each beat, the hands throw alternately to a steady rhythm and the hands catch and throw in the normal fashion. We shall also assume that the dwell ratio, defined as the proportion of time that each hand is full, is $\frac{1}{2}$. We shall talk about balls, although everything applies to clubs and rings and whatever else you care to juggle. We shall limit ourselves to patterns which use a finite number of balls and repeat in a finite number of beats.

1 Air Time and Cycle Time

We define the **air time** of a throw to be the number of beats the ball spends in the air and the **cycle time** to be the number of beats until the ball is thrown again. The restrictions we have imposed mean that when a ball is caught it will be thrown again one beat later, so the cycle time is always one more than the air time. In the most basic three-ball pattern, a three-ball cascade, every throw has an air time of 2 beats and a cycle time of 3 beats.

We can consider the most basic pattern with n balls, called the **ground state pattern**, to be the pattern where each throw has cycle time n. Since all the throws have the same air time the balls must stay in the same order, e.g. for $n = 4$ the balls are thrown in this order: $ABCDABC\dots$. As the hands are throwing alternately it is clear that for even n the throws do not cross from one hand to the other and for odd n they do. The ground state pattern is called a **fountain** if n is even and a **cascade** if n is odd.

A simple piece of dynamics gives the time spent in the air by a ball thrown to a height h above the juggler's hands as $2\sqrt{\frac{2h}{g}}$. So if we fix a rhythm then the height of each throw is determined by its air time and hence by its cycle time.

The siteswap notation of a pattern is the minimum non-repeating sequence of numbers giving the cycle time of each throw. The cycle times of throws in the three ball cascade is $3333\dots$, so the siteswap notation is simply 3. Similarly the basic pattern with n balls is written n. By convention there are no spaces between the numbers and you use a for 10, b for 11, etc.. The length of the siteswap notation for a pattern tells you how many throws it takes before the pattern starts to repeat.

It is interesting to consider the ground state juggling patterns with one or two balls. With two balls every throw in the basic pattern has cycle time 2. But if the ball is going to be thrown again in 2 beats' time, which is when that hand throws next, it does not need to be thrown at all: you can just keep hold of it. Hence when we have to do a throw with cycle time 2 we can just hold on to the ball, and the basic pattern with two balls is just holding both of them. With one ball, every throw of the ground state pattern has cycle time 1 and air time 0. If the ball has to be thrown on the next beat with the other hand then it must be passed across to that hand. So a throw with cycle time 1 is a quick pass across and the basic pattern with one ball is to throw it from hand to hand.

We still have a slight problem with throws with cycle time 0. They should have air time -1, which does not sound very promising. Actually they just represent an empty hand for one beat. The negative air time comes from the assumption that the hand has a ball in it ready to throw. An alternative interpretation is that the air time of -1 represents a ball travelling backwards in time, an anti-ball if you will, whose addition to the pattern cancels out the ball assumed to be in the hand, resulting in an empty hand.

Many people have written siteswap simulators which animate a stick man juggling any siteswap you enter (see, for example, [8]). Unless you have quite a lot of experience with siteswaps they are hard to visualise, so I recommend you use a siteswap generator to see what the patterns listed look like.

2 What is a Valid Siteswap?

We have defined how to write any pattern which satisfies our criteria in siteswap notation as a series of numbers. If we want to invent new patterns we must now consider when a series of numbers is a valid siteswap, i.e., it represents a pattern which can be juggled. Let us consider the siteswap notation for some easy patterns.

One Ball

1	pass the ball from hand to hand (one ball cascade)
20	hold the ball in one hand (one ball out of the two ball fountain)
300	throw the ball from hand to hand (one ball from the three ball cascade)

Two Balls

2	hold one ball in each hand (two ball fountain)
31	throw one ball across, pass the other underneath it (two ball shower)
40	juggle two balls in one hand, nothing in the other hand

Three Balls

3	three ball cascade
42	juggle two balls in one hand, holding one ball in the other
51	one hand throws high crossing throws, the other passes across (three ball shower)
423	an easy pattern
441	a slightly harder pattern

All the series listed share the common property that the mean of all the numbers in the siteswap is the number of balls used. It can be shown that this is true for all juggling patterns, but not every such series of numbers is a valid juggling pattern. We need a stronger condition which ensures that two balls never land at the same time: a sequence $a_1 \cdots a_n$ is a valid siteswap if and only if

$$a_i + i \not\equiv a_j + j \pmod{n} \ \forall\, i \neq j.$$

This is stronger than $(\sum_{i=1}^{n} a_i)/n \in \mathbb{N}$: it implies that the $a_i + i \pmod n$ must take each value in $\{1, 2, \cdots, n\}$ precisely once, so

$$\sum_{i=1}^{n}(a_i + i) \equiv \sum_{i=1}^{n} i \pmod n$$

$$\Rightarrow \qquad \sum_{i=1}^{n} a_i \equiv 0 \pmod n.$$

So n must divide $\sum_{i=1}^{n} a_i$, and hence $(\sum_{i=1}^{n} a_i)/n \in \mathbb{N}$.

To see the motivation behind this condition, observe that $a_i + i$ is precisely the number of beats after starting that the i^{th} ball thrown will be thrown again. Since we have assumed a dwell ratio of $\frac{1}{2}$, the i^{th} ball thrown will land $a_i + i - 1$ beats after starting. So if no two $a_i + i$ are congruent modulo n, then no two $a_i + i - 1$ are equivalent modulo n, and hence no two throws will land at the same time.

This allows us to create new patterns. Who would have guessed that 12345, 7531 and 83831818131881317163l are all juggling patterns? Beyond the novelty of having more patterns to try, we can look for patterns with certain properties:

1. 5551 is a four ball pattern created using the ideas which led to siteswap notation from the sequence of four ball patterns 4, 53, 552. It is a good stepping stone to learning the five ball cascade as most of the throws are 5s and none of the throws is a 2 or a 0 so there are no beats on which you don't throw anything.

2. 66161 and 66661 are four and five ball patterns which are very good practice for learning the six ball fountain.

3 Generalisations of Vanilla Siteswap

The notation system I have just outlined is often called **vanilla siteswap** as several other 'flavours' exist. These are generalisations of siteswap in which some of the restrictions are relaxed. The simplest allows a hand to throw more than one ball at the same time, to different heights (called a **multiplex** throw). Another, **synchronous siteswap**, allows both hands to throw at the same time. Others have been designed for patterns shared between two or more people and to specify the position of the hands for each throw and catch. In fact siteswap works in full generality for a juggler with any number of hands, although this is rarely useful.

4 States

States are another way of looking at juggling tricks, and are perhaps the easiest route to proving what I have stated above. We define the **state** of a juggling pattern at a given time to be a finite series of 0s and 1s, where a 0 in the n^{th} place means no ball has yet been thrown which is going to land in n beats' time and a 1 means there is such a ball. Hence for a state $s \in \{0, 1\}^r$ the number of balls b is the number of 1s in the state: $b = \sum_{i=1}^{r} s_i$.

To ensure that the state need only be a finite series we choose a positive integer n and consider only patterns with siteswap values no higher than n. Having a maximum throw value n means there are a finite number ${}^nC_b$ of states for b balls. We call this the **maximum**

throw height, although strictly speaking it is not a height but the number of beats until the ball is thrown again.

We can now consider the **state graph** where the states form the vertices and there is a directed edge from state ϕ to state θ when there is a throw which takes you from state ϕ to state θ. A valid pattern is a circuit in this juggling state graph which includes at least one vertex and one edge. By specifying a maximum throw height we have restricted ourselves to a finite portion of the infinite graph of all possible juggling states with a given number of balls.

To find the resultant state given an initial state and the siteswap value m for a throw you need first to make sure the throw is valid from the initial state:

- If $s_1 = 0$ then $m = 0$ is the only valid throw value as no balls are landing for you to throw.
- If $s_1 = 1$ then you must have $m > 0$ as you have to throw the ball which is landing.
- You must have that the $(m+1)^{\text{th}}$ digit of the initial state is 0 or $m = r$, as you can't have two balls landing at the same time.

If these conditions hold then to get the resultant state you remove the first digit, add a 0 on the end and turn the m^{th} digit into a 1.

Examples with three balls and throws no higher than 5:

- The three ball cascade stays in the state 11100 by the edge representing a 3. We call this the ground state, because it is the state which the basic pattern stays in.
- The three ball shower 51 moves between the state 11010 and 10101 by the edge representing a 5 and the edge representing a 1.
- 441 goes from 11100 to 11010 to 10110 and then repeats.

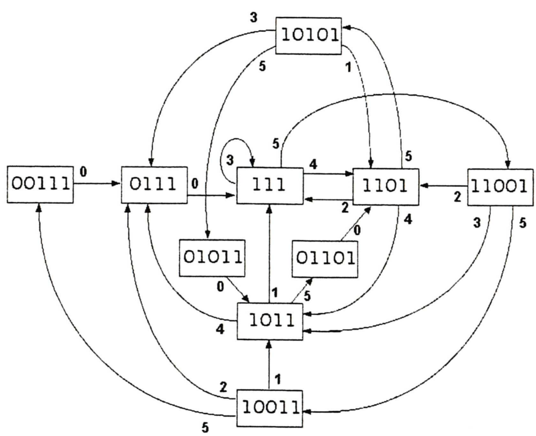


Figure 1: The state graph for three balls and maximum throw height 5. Note that trailing zeros have been truncated, so the state 11100 is written 111. Figure courtesy of Ben Beever.

5 Ground State and Excited State Patterns

If a pattern can be juggled straight from the basic pattern for a given number of balls, then the path through the graph of states which represents those patterns must pass through the ground state. These patterns are called **ground state patterns**. It is easy to change between the basic pattern and ground state patterns, and hence between ground state patterns.

Patterns which do not pass through the ground state are called **excited state patterns** and they are written in the form 4 51 2 where 51 is the pattern (in this case the three ball shower), 4 is a sequence of throws which get you from the basic pattern to a state where you can start juggling the pattern, and 2 is a sequence of throws which takes you from the pattern back to the basic pattern. There are a countably infinite number of possible entry and exit sequences but in practice you are only interested in either the shortest or the easiest to juggle (often these are the same). In the case of the three ball shower there is another entrance sequence, 52, which is more common because the 2 makes it easier to juggle. Note that 4512 and 52512 are both valid siteswaps: they correspond to continually switching between the three ball shower and the three ball cascade using different entrance sequences. This allows you to check whether an excited pattern can be juggled: 1 51 2 is clearly not valid because $(1+5+1+2)/4 = 9/4$ which isn't an integer; 25 51 2 is not valid because although the mean of the numbers is an integer, two throws land at the same time: $2+1 \equiv 5+3 \pmod 5$.

6 Prime patterns

A pattern is called **prime** if it never goes through the same state twice in one round of the pattern. This is equivalent to saying that the pattern cannot be split up into two valid patterns with the same number of balls as the original pattern. For example, 534 (four balls) is not prime as both 53 and 4 are valid siteswaps with four balls. 53 is prime even though 5 and 3 are valid patterns, as they use five and three balls respectively.

The siteswap pattern 838318181318813171631 is a period 21, prime, four ball siteswap with maximum throw height 8, which was chosen to have no 2s or 0s (because they would make it easier to juggle). The longest prime pattern with four balls and a maximum throw height of 8 has period 58 – in fact there are 44 of them!

7 Historical Development

For a long time there was no good way of notating a juggling trick, short of a verbal description. Siteswap was simultaneously invented around 1985 by three independent groups, though each group had a completely different interpretation of what was going on. The approach given above, which has become the accepted explanation, draws on the work of the Cambridge group, comprising Mike Day, Colin Wright, Adam Chalcraft and James Mellor, who were mainly mathematicians. They started from ladder diagrams, which are space-time diagrams showing the position of the balls as seen from above (so the height is ignored) as time progresses.

The other two groups both hailed from California: Bruce Tiemann from Caltech and Paul Klimek from Santa Cruz. It appears that Paul Klimek was the first, and he was also the only one to notice the distinction between ground state and excited state siteswaps – indeed he coined those terms. Bruce Tiemann was thinking about siteswaps as permutations of the landing times of the balls – which is why he came up with the name siteswap (the Cambridge

group called it the Cambridge notation). He also used ladder diagrams to keep track of w
throws he had to make.

In 1990 Jack Boyce, also from Caltech, invented the state model for juggling pattern
well as adapting siteswap to allow throwing more than one ball from a hand (multiplexi
throwing with both hands at once (synchronous patterns) and throwing balls between tw
more people (passing patterns).

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[9] `http://www.jugglingdb.com/`
The International Juggling Database, contains lots of helpful information.

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Siteswap Ben's Guide to Juggling Patterns: a book written by Ben Beever, one of th best siteswap jugglers in the world. There are sections aimed at mathematicians.

[11] `http://www.jugglingdb.com/articles/index.php?id=59`
An article on Longest Prime Siteswaps by Jack Boyce, which goes into more detail abou the state model for juggling patterns.

Invertible Set Theory

David Chow

Remarkably, it is sometimes possible to come across mathematics which makes sense when read upside down! The problem of finding such statements arose recently in [1]. We shall investigate some results of this type arising from considering finite sets. The results are non-trivial when read both normally and inverted, and the inverted equations are distinct from the originals.

Suppose that H, S, X and Z are finite sets; we denote the cardinality of X by $|X|$. It is well known that

$$|X \cup Z| + |X \cap Z| = |X| + |Z|, \tag{1}$$

and hence

$$|S \cup X| + |X \cup Z| + |Z \cup S| + |S \cap X| + |X \cap Z| + |Z \cap S| = |S| + |S| + |X| + |X| + |Z| + |Z|. \tag{2}$$

Then, from the inclusion-exclusion principle, we have

$$|S \cap X \cap Z| + |S \cup X| + |X \cup Z| + |Z \cup S| = |S \cup X \cup Z| + |S| + |X| + |Z|. \tag{3}$$

Another well-known result is

$$S \cap (X \cup Z) = (S \cap X) \cup (S \cap Z), \tag{4}$$

from which more complicated expressions can be derived, such as

$$(H \cup S) \cap (X \cup Z) = [(H \cap X) \cup (S \cap X)] \cup [(H \cap Z) \cup (S \cap Z)]. \tag{5}$$

Although there are many individual symbols which have some meaning when viewed upside down, very few appear to make sense with the adjoining symbols when inverted. The very shortest statements such as $d + p = p + d$ are unsurprisingly trivial and not so interesting. It is often the case that the upside down symbol is too closely related to the original, such as being the reverse or even identical, to give two distinct statements. Therefore the restriction on a mathematical statement that it makes sense when viewed upside down, so that both statements are non-trivial and distinct, seems to be rather severe; it would be amusing to discover more examples.

I would like to thank D. Collier and D. Hodge for a useful remark.

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The Physics of Floating

Dominic Vella

1 Introduction

It seems to me highly inappropriate that a subject which excited Archimedes to the p where he supposedly felt compelled to cry "Eureka!" and exhibit himself through the str of Syracuse appears never to have been the subject of an article in *Eureka*, the journa the Archimedeans. Perhaps this has been justified on the grounds that Archimedes' Princ is so simple that we were all convinced in primary school that our teacher had taugh everything there could possibly be to know about it. On the contrary, there remains muc discover if only we are prepared to focus at the scale of bubbles floating in wine and break cereal floating in milk rather than an ancient Greek getting into his bath. This artic aimed at redressing this imbalance by reviewing what we might all have learnt a long t ago were we a much smaller pond-surface-dwelling species. At such a scale, a new force wo enter our lives: surface tension. Of course surface tension is not really new to us, but s of its effects certainly would be: objects much denser than water are able to remain floa (provided that they are small enough) and small objects that are less than a few millime apart are subject to an exotic force that cause them to clump together with similar obje

For most pond-walking creatures, surface tension is what prevents them from drown since it provides a restoring force sufficient to overcome the animals' weights, allowing t to remain at the interface between air and water. The vertical deformation $z = h(x, y)$ o interface from the flat due to the presence of such an object is determined by the requiren that the pressure drop across a deformed interface (proportional to the curvature of deformation) must counteract the hydrostatic pressure brought about by that displacem Mathematically, this is expressed by the Laplace-Young equation

$$\gamma \nabla \cdot \mathbf{n} = -\rho g h,$$

where $\mathbf{n}(x, y)$ is the unit normal to the interface, γ the surface tension coefficient of liquid-gas interface, ρ the density of the liquid and g the acceleration due to gravity. U $\mathbf{n} = \nabla(z-h)/|\nabla(z-h)|$ with all lengths non-dimensionalised by the *capillary length*, L $\sqrt{\gamma/\rho g}$, and writing $H = h/L_c$, equation (1) takes the form

$$\frac{\nabla^2 H}{(1 + H'^2)^{3/2}} = H$$

where $H = h/L_c$ and (2) is to be solved with the boundary conditions that $H(\pm\infty)$ and that the angle that the meniscus makes to the solid boundary is some fixed constan known as the contact angle.

2 A generalisation of Archimedes' Principle

If we imagine ourselves in the miniature realm of the pond skater, we realise that a floa object displaces liquid not only because of the excluded volume effect (with which we familiar from school) but also because of the interfacial deformation that it causes elsewh What would Archimedes have made of this situation?

Keller [2] and Mansfield *et al.* [3] have considered this question only surprisingly recently and have shown that the vertical force on an interfacial object is equal to the weight of the total volume of liquid displaced by the object, including that displaced in the meniscus away from the body itself. Here we shall content ourselves with dividing the total force on the floating object into two components: $\mathbf{F}_{st}$, coming from the surface tension force acting at the contact line where the gas, liquid and solid meet, and $\mathbf{F}_{hp}$, which results from the action of the (hydrostatic) pressure on the parts of the object's surface that are wet. The first of these gives a force per unit length of the contact line equal to γ in the direction tangent to the interface itself, while the second is given by

$$\mathbf{F}_{hp} = \int_S \rho g z \mathbf{N}\, \mathrm{d}A, \tag{3}$$

where S is the surface wetted by the liquid, $\mathbf{N}$ is the normal pointing from that surface into the liquid and $\mathrm{d}A$ is an area element on the surface. The vertical component of this force is given by

$$\hat{\mathbf{e}}_z \cdot \mathbf{F}_{hp} = \int_S \rho g z \hat{\mathbf{e}}_z \cdot \mathbf{N}\, \mathrm{d}A = \int_A \rho g z\, \mathrm{d}x\, \mathrm{d}y, \tag{4}$$

where we have used the result that $\hat{\mathbf{e}}_z \cdot \mathbf{N}\, \mathrm{d}A$ is the projection of the surface S onto the x-y plane, denoted by A. Physically, the r.h.s. of (4) corresponds to the weight of liquid displaced between the wetted region of the object and the x-y plane tangent to the undeformed interface.

3 Floating versus sinking

As a simple application of Archimedes' Principle with the additional complication of surface tension, we next consider a question of considerable importance to creatures that live on the surfaces of ponds everywhere, namely "how heavy can an object of a given size be before it will sink?". Clearly, the detailed morphology of the object will be an important factor in reality, but here we consider a toy problem that is easily tractable at the expense of being highly idealised. We consider a single cylinder of infinite length lying horizontally at a liquid-gas interface as represented in Figure 1. The cylinder has a density ρ_s, which enters the calculation only through $D = \rho_s/\rho$, its value relative to the liquid density ρ. The other parameters of interest are the contact angle θ, which is a chemical property of the three phases that meet at the contact line, and the radius of the cylinder, R. The radius R is non-dimensionalised by the capillary length L_c and is represented by the *Bond number* $B = R^2/L_c^2$ for historical reasons. For small Bond numbers the effects of surface tension are important, but for large Bond numbers (a radius of a few centimetres or more for an air-water interface) the radius is so large that surface tension effects are negligible.

We shall determine the maximum density ratio, $D_{\max}$, that a cylinder with a given Bond number and contact angle can have before it sinks, but before embarking on the detailed calculation we investigate the limits $B \ll 1$ and $B \gg 1$ physically. In the latter case, we expect the influence of surface tension to be negligible and so only objects with density less than or equal to the density of the liquid can float, i.e., for $B \gg 1$

$$D_{\max} \sim 1. \tag{5}$$

For small objects, we expect the Archimedean buoyancy to be negligible and the object to float purely by virtue of the vertical component of surface tension at the contact line. Per unit length, the maximum force that can be generated via this mechanism is 2γ, which must

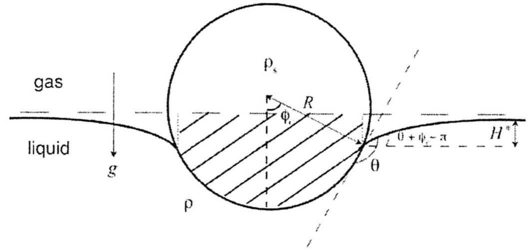

Figure 1: Setup and notation for a circular cylinder floating horizontally at the interfa between a liquid and gas. The hatched area represents the displaced liquid whose weight equivalent to the buoyancy force due to hydrostatic pressure on the cylinder.

balance the weight of the cylinder, namely $\rho_s g \pi R^2$. This simple argument gives that f $B \ll 1$

$$D_{\text{max}} \sim \frac{2}{\pi B}.$$

Having determined the asymptotic behaviour of D_{max} for extreme values of B it remains calculate its behaviour for intermediate B. Mansfield *et al.* [3] have shown that it is possib to provide a lower bound on D_{max} by approximating the interfacial gradients as being sma For a full treatment of this problem, however, it is not enough to make such an assumptio since just before sinking we should expect the interface to be subject to large deformatio beyond the regime of this linear theory. In the slightly contrived geometry chosen here, are able to make progress without this assumption since the Laplace-Young equation (2) f the interfacial deformation takes the relatively simple form

$$H = \frac{H''}{(1+H'^2)^{3/2}},$$

which can be integrated once subject to the requirement that $H(\pm\infty) = H'(\pm\infty) = 0$ to gi

$$H^2 = 2\left(1 - (1+H'^2)^{-1/2}\right).$$

Using (8) and the other boundary condition that the interface makes an angle $\theta + \phi_c -$ with the horizontal at the contact line, the interfacial deformation at the contact line, H^*,

$$|H^*| = \sqrt{2(1 - |\cos(\theta + \phi_c)|)},$$

where ϕ_c is as defined in Figure 1 and we have had to be careful in choosing the corre branch of the cosine. With this result, the vertical force balance condition becomes (in no dimensional terms)

$$f(\phi_c) \equiv 2\sin\phi_c\sqrt{2B(1 - |\cos(\theta + \phi_c)|)} + B(\phi_c - \sin\phi_c\cos\phi_c) - 2\sin(\theta + \phi_c) = \pi B D.$$

Here, the r.h.s. of (10) is the non-dimensional weight (per unit length) of the cylinder, whic must be balanced by the weight of displaced liquid in the hatched area of Figure 1 (the fir two terms on the l.h.s.) and the vertical component of the surface tension force (the thi term on the l.h.s.). To determine the maximum density that a cylinder may have witho sinking, we must find the maximum value of $f(\phi_c)$ for a given value of θ and B. Unfortunatel

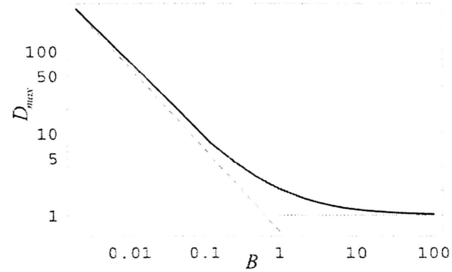

Figure 2: Solid line shows the numerically computed maximum density, $D_{\max}$, of a floating cylinder with contact angle $\theta = 2\pi/3$ as a function of the Bond number, B. The short and long dashed lines represent the asymptotic results (5) and (6) respectively.

This can only be done numerically, with the results for $\theta = 2\pi/3$ and different cylinder Bond numbers plotted in Figure 2 along with the asymptotic results (5) and (6). The asymptotic results are indeed borne out by the numerical calculation for extreme values of B, but for $B \in (0.1, 10)$ we see that the combined effects of surface tension and hydrostatic pressure are enough to support considerably denser objects than either could alone.

There appears to be little experimental data to compare these results with – particularly for intermediate Bond numbers. However, Gao and Jiang [1] recently measured the vertical force that can be supplied by a water strider's leg before it sinks. In these experiments, $B \approx 0.01$ and the leg length is around 9mm so that the theory above predicts a vertical force of around 143 dynes, which compares reasonably favourably with the 152 dynes measured experimentally.

4 How (not) to walk on water

Having seen that for objects of very small Bond number the weight that can be supported is proportional to the wetted perimeter of the floating object, it is natural to consider whether this can be used to circumvent the normal physics that prevents humans from walking on water. In particular, it seems possible that by using special shoes with a fractal outline, the Koch snowflake, for example, one might be able to make shoes that would allow us to perform such a feat. Before you rush off to try this, however, bear in mind the generalisation of Archimedes' Principle discussed earlier: the maximum upward force that can be produced is equal to the *total* weight of liquid that is displaced (including that within the interfacial menisci). Since the density of humans is roughly that of water, fractal shoes with a cross-sectional area typical of ordinary shoes would thus have to displace a volume of water roughly equal to the volume of the human they are supposed to support. Since interfacial deformations extend away from the object only a few times the capillary length and can only extend to depths of the same order at the edge of the floating object, that volume would predominantly come from the depth to which the shoes would fall (without sinking!), requiring a very odd shoe design and the wearer only just to have their head above the normal surface of the water, if at all. If you wish to try your luck making such a pair of shoes, then you have been warned – you may well be able to stay dry. . . just.

References

[1] X. Gao and L. Jiang, "Water-repellent legs of water striders", *Nature* **432**, (2004) 36.

[2] J.B. Keller, "Surface tension force on a partly submerged body", *Phys. Fluids* **10** (1998) 3009–3010.

[3] E.H. Mansfield, H.R. Sepangi and E.A. Eastwood, "Equilibrium and mutual attraction or repulsion of objects supported by surface tension", *Phil. Trans. R. Soc. Lond.* A **355** (1997) 869–919.

Ed: The Puzzle Hunt was held in December as a prelude to the Archimedeans' festive Christmas Party. One of the tasks allotted to the Hunters was to produce a story of at least 100e words, on a mathematical subject, containing every letter of the alphabet and ending, as good stories occasionally do, with a moral. The following Borgesian triumph didn't quite achieve the target word count and does not appear to contain the letter j; its author was nevertheless pronounced the winner.

Puzzle Hunt

Alexander Shannon

When once perusing the excellent selection of mathematics works in my College library, I suddenly had an inexplicable whim to follow up a reference I had seen the previous day. It concerned Zorn's Lemma, an amazingly useful result but one very much associated with the axiom of choice, and hence not accepted by anyone not prepared to be sufficiently cavalier about their attitude to things infinite. Naturally, containing such sensitive material, the reference itself was chained to one of the shelves with a suitable warning attached. I had soon found the answer to my query, but continued to leaf through the pages as they proved result after result on sets – partially ordered ones, chain complete partially ordered ones, and ones with other fascinating names so long I cannot recall them. However, as I carried on reading a strange phenomenon occurred – I seemed not to be getting any further through the book. As I passed chapters talking about transfinite ordinals, transfinite cardinals, alephs, omegas, epsilons, I noticed the light fading at the windows. Not knowing what the time was, and becoming ever more curious about this strange phenomenon, I looked to see what was the number of the last page of the book. Days I spent searching before I thought to look back at the chain holding the book to the shelf, only to notice that it had a link missing. . . .
The moral is: *always check for chain-completeness before searching for a maximal element!*

A purely analytic irrationality proof

Tom Müller

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Irrational numbers are of common interest in all mathematics and especially in real and complex analysis. Most books and lecture notes on analysis prove the irrationality of $\sqrt{2}$ (or more generally the irrationality of non-integer solutions to an equation which can be written $x^2 - b = 0$ for some natural b) following the classical argument said to be discovered by Pythagoras (compare Heath [2], 90 ff.). This proof makes use of the Fundamental Theorem of Arithmetic, which states that every natural number except 1 can be written in an essentially unique way as the product of prime numbers. The purpose of this article is to present an alternative irrationality proof using only basic analytical methods to reach the same goal.

Theorem: *Let ϕ be a non-integer solution of an equation of the form*

$$x^2 + ax + b = 0$$

with integer coefficients a and b. Then ϕ is irrational.

Proof: As ϕ is not an integer, we have $0 < \phi - \lfloor\phi\rfloor =: \rho < 1$, where $\lfloor y\rfloor$ is the floor function of y. This means that

$$\rho^n \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{1}$$

We shall now prove by induction that for every n there are integers a_n and b_n such that $0 < \rho^n = a_n + b_n\rho$. For $n = 1$ this is clear, and for $n = 2$ we obtain, with $\phi^2 = -a\phi - b$,

$$\begin{aligned} \rho^2 &= \phi^2 - 2\lfloor\phi\rfloor\phi + \lfloor\phi\rfloor^2 \\ &= (-a\phi - b) - 2\lfloor\phi\rfloor\phi + \lfloor\phi\rfloor^2 \\ &= (-a - 2\lfloor\phi\rfloor)\rho + (-b - a\lfloor\phi\rfloor - \lfloor\phi\rfloor^2), \end{aligned}$$

from which the desired result follows by defining $a_2 := -b - a\lfloor\phi\rfloor - \lfloor\phi\rfloor^2$ and $b_2 := -a - 2\lfloor\phi\rfloor$. Now suppose the claim is true for $n - 1$ with $n \geq 3$. Then

$$\begin{aligned} \rho^n &= (a_{n-1} + b_{n-1}\rho)\rho \\ &= (a_{n-1} + b_2 b_{n-1})\rho + a_2 b_{n-1}, \end{aligned}$$

using the inductive hypothesis for ρ^{n-1} and ρ^2. Taking $a_n := a_2 b_{n-1}$ and $b_n := a_{n-1} + b_2 b_{n-1}$ proves the claim for ρ^n.

Considering the preceding result and (1) leads to the inequality

$$0 < a_n + b_n\rho = \rho^n \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty \qquad \forall n \in \mathbb{N}, \tag{2}$$

which already implies the irrationality of ρ and of course that of ϕ as well; if we suppose that ρ is rational there would be a natural number k such that $k\rho$ and therefore $k(a_n + b_n\rho)$ are integers for all n. But because $\rho^n \longrightarrow 0$ as $n \longrightarrow \infty$ we can choose n large enough that ρ^n is smaller than $\frac{1}{k} > 0$, with the consequence that our inequality (2) would imply the existence of an integer falling strictly between 0 and 1, which of course is impossible. This contradiction completes the proof.

Remark: The argument can be generalized to prove a result known to Gauss [1] on the irrationality of every non-integer solution of equations of the type

$$x^m + \sum_{\nu=0}^{m-1} c_\nu x^\nu = 0, \tag{3}$$

with integer coefficients c_ν and $1 < m \in \mathbb{N}$. Let ϕ be such a solution of (3). Defining $\rho := \phi - \lfloor\phi\rfloor > 0$ again leads to $\rho^n \to 0$ as $n \to \infty$. Using the binomial formula and the expression $\varphi^m = -(c_{m-1}\phi^{m-1} + \ldots + c_1\phi + c_0)$ we see that ρ^m can be brought into the form $a_0^{(m)} + a_1^{(m)}\rho + \ldots + a_{m-1}^{(m)}\rho^{m-1}$ with integer coefficients $a_\nu^{(m)}$. Then, again by induction, we get for all $n \geq m$

$$\rho^n = a_0^{(n)} + a_1^{(n)}\rho + \ldots + a_{m-1}^{(n)}\rho^{m-1} \tag{4}$$

for some integers $a_\nu^{(n)}$. Hence the inequality

$$0 < k^m \left(a_0^{(n)} + \ldots + a_{m-1}^{(n)}\rho^{m-1}\right) = k^m\rho^n \to 0 \qquad \text{as } n \to \infty \tag{5}$$

holds for all $n \geq m$ and for all natural k. This immediately implies the irrationality of ρ and of ϕ.

References

[1] C.F. Gauss, *Disquisitiones Arithmeticae* (1801) (English translation by A.A. Clarke, Yale University Press, 1966) §11.

[2] T. Heath, *A History of Greek Mathematics, Volume I: From Thales to Euclid*, Dover (1981) (republication).

(8) – A Cryptic Clerihew

d'Ag

Charles' mother
And my sister's brother
Are bound by the head
With an old piece of thread.

Thoughts on Geometric Realisability

Jonny Evans

The problem of embedding topological spaces in an ambient Euclidean space naturally leads one to consider geometric realisations of abstract simplicial complexes. An abstract simplicial complex is a combinatorial object with which one replaces a space. Explicitly, it is a set, V, of vertices and a collection, S, of finite subsets of V such that if $s \in S$ then any subset of s is also in S. How does this correspond to the original space, X? If X had a triangulation, then one could take V to be the set of vertices in the triangulation and require $s \in S$ if and only if s consists of vertices of an edge, face, etc. in the triangulation.

Why did I use the qualifier "abstract"? A (non-abstract!) simplicial complex is defined to be a collection of simplices (like n-dimensional tetrahedra) in $\mathbb{R}^m$ which fit together nicely (i.e., any two intersect in a face). Such a space has an obvious triangulation and is therefore naturally associated with an abstract simplicial complex, K. It is said to be a geometric realisation for K. It is equally obvious that a given abstract simplicial complex has (uncountably) many geometric realisations, but that all are isomorphic in a sense I will now define.

A simplicial map is a piecewise linear map between (non-abstract) simplicial complexes which takes simplices to simplices. Let $\mathbb{S}$ denote the category of simplicial maps between simplicial complexes. Then by isomorphism I mean an isomorphism in this category.

To return to the problem of embedding topological spaces in $\mathbb{R}^n$, if the space in question is triangulable this reduces to the problem of finding a geometric realisation in $\mathbb{R}^n$ for the abstract simplicial complex of the triangulation. At first sight this may seem hopeless: some triangulations involve many vertices and it's not obvious at first that it will be possible to geometrically realise them in a low-dimensional space. But an amazing fact about simplicial complexes is that one can find a dimension in which they have a realisation which depends only on the dimension of the highest-dimensional subsimplex, as the following elegant proof [1] shows. A set of points in $\mathbb{R}^n$ is called affinely independent if, when we consider them as lying in an affine hyperplane of $\mathbb{R}^{n+1}$, the position vectors of any n as points in $\mathbb{R}^{n+1}$ are linearly independent. If you think about it, any simplicial n-complex drawn on such a set of points will be such that no two subsimplices intersect one another in their interiors.

Theorem 1 *Every simplicial n-complex has a geometric realisation in $2n+1$ dimensions.*

Proof This is equivalent to saying that for arbitrary N we can find a set of N points in $\mathbb{R}^{2n+1}$ such that any $2n+2$ of them are affinely independent. Such a set of points is provided by $\{(i, i^2, \ldots, i^{2n+1}) : i \in \{1, 2, \ldots N\}\}$, as we can see from an elementary application of the Vandermonde determinant:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ i_1 & i_2 & \cdots & i_{2n+2} \\ i_1^2 & i_2^2 & & i_{2n+2}^2 \\ \vdots & & & \vdots \\ i_1^{2n+1} & i_2^{2n+1} & \cdots & i_{2n+2}^{2n+1} \end{pmatrix}$$

The subset of points is affinely dependent if and only if this vanishes, if and only if two of the i_j coincide. □

Now comes my question. Two supervisors have advised me that it probably does not have an answer (at any rate not a simple one) and I agree, but I think it is an interesting question nonetheless.

Let $m(K)$ denote the minimal dimension for which a simplicial complex K has a geometri realisation in $\mathbb{R}^{m(K)}$. Suppose that K' were another simplicial complex and that there existe an injective simplicial map $\iota : K' \hookrightarrow K$. Then K' would also have a geometric realisation i $\mathbb{R}^{m(K)}$, so $m(K)$ is an (extremely coarse) invariant under isomorphisms in the category $\mathbb{S}$.

How is this related to other invariants like homology? Is there an algorithm for decidin whether a given simplicial complex is realisable in a given dimension?

Why might one expect there to be such a relationship? A simple case: orientable close compact 2-manifolds can be embedded in $\mathbb{R}^3$, but non-orientable ones require an ambien $\mathbb{R}^4$. And how do we distinguish orientable and non-orientable 2-manifolds? By their secon homology group, which is $\mathbb{Z}$ for the orientable case and trivial otherwise.

Interesting things happen when we look at embeddings of spaces in higher-than-minima dimensions. For instance, S^1 can be embedded in $\mathbb{R}^2$. All such embeddings are essentiall the same because of the Jordan curve theorem. However, in $\mathbb{R}^3$ there are interesting tangle embeddings called knots. In codimension 3 and higher, these knots can be untangled. I the same way, spheres can be knotted in $\mathbb{R}^4$. In fact, tori can be knotted in $\mathbb{R}^3$, as can b seen by taking the boundary of a tubular neighbourhood of a trefoil knot. Embeddings o tori in three-dimensional space are important in the study of 3-manifolds, because of th Jaco-Shalen-Johannson Theorem [3] which allows us to decompose a 3-manifold by cuttin along embedded tori. Another, less obvious, example of a complex which can be knotted i the space of its minimal embedding dimension is the Klein bottle, which can be knotted i $\mathbb{R}^4$ [2].

I am very grateful to Dr J. Woolf for the excellent suggestions he made in proof-reading thi article.

References

[1] P.J. Hilton and S. Wylie, *Homology Theory*, Cambridge University Press (1962).

[2] K. Yoshikawa, "The order of a meridian of a knotted Klein bottle", *Proc. Am. Math Soc.*, **126** (1998) 3727–3731.

[3] A. Hatcher, "Basic Topology of 3-Manifolds", notes on the internet a `http://www.math.cornell.edu/~hatcher/` (unpublished in hard copy).

Book Reviews

Modern Dynamical Systems and Applications

Edited by **M. Brin, B. Hasselblatt and Y. Pesin,** Reviewed by **Dan Jane**

Cambridge University Press, 2004,
ISBN 0-521-84073-2.

Dynamical Systems began with the study of repeatedly composing a suitable map many times. Now enriched with measure theory, manifold structures, etc., the subject has many fascinating problems and constructs at all levels of mathematics. Here the authors have brought together a very comprehensive collection of papers from across the field. Overall, the papers themselves are well written and it is reassuring to see that a large minority have already made names for themselves as clear and engaging writers. Personally, I very much enjoyed the selection they have chosen, even though many would not have been my usual reading material. However, it is important to stress that these are cutting edge papers and all assume at least some knowledge of rather advanced topics — from Riemannian Geometry to Statistics (through, alas, Algebraic Topology!). The most accessible papers are by Tabachnikov and Eberlein on Magnetic Billiards and 2-Step Nilpotent Lie Groups respectively. These help convey an idea of the enormous variety of subjects that the theory can shed new light on. With an enquiring nature, a few friends and a handy geometer to e-mail these papers would be a good introduction to the exciting developments at one of the many frontiers of modern mathematics.

Knots and Links

By **P.R. Cromwell,** Reviewed by **David Chow**

Cambridge University Press, 2004,
ISBN 0-521-83947-5.

The idea of a knot is easily visualized and is something anybody on the street can easily grasp. In addition, knots have a surprisingly large range of applications, from quantum field theory to molecular biology. Unfortunately, the mathematical theory is deep and so usually taught only at postgraduate level. This book provides a modern, readable, introductory text which makes the subject accessible to advanced undergraduates who would otherwise likely not see any of the theory. There are few prerequisites — only basic mathematics which all undergraduates will have covered before their final year and, in particular, no algebraic topology is assumed. Standard topics are dealt with starting from scratch, including Seifert matrices and polynomial invariants. Other topics are skewed towards the author's own interests with a geometrical and combinatorial flavour, such as surface intersections, "cut and paste" surgery and properties of tangles. The text is clearly written, in a pedagogical style, and beautifully illustrated by countless diagrams. Plenty of examples and exercises are provided. Usefully, for the intrepid reader wishing to delve further, there are a "few external references", in fact 255 of them! I recommend this book to anyone looking for a good, basic introduction to knot theory.

Practical Applied Mathematics: Modelling, Analysis, Approximation

By **S. Howison,** Reviewed by **Erica Thompsor**

Cambridge University Press, 2005.
ISBN 0-521-60369-2.

One of the highly-regarded series of Cambridge Texts in Applied Mathematics, this bool presents a solid introduction to various areas of applied mathematics at the level of secon and third year undergraduate courses in Mathematics and Natural Sciences. The emphasis i very much upon applications and as such each section includes a diverse range of case studie and a large number of relevant exercises. The informal style is generally a strength of th book; the author writes clearly and well, and his treatment of the material is a good deal les dry than many others. The notes in the margin can seem somewhat patronising on occasion but the majority are a useful addition to the text. Little background is assumed beyond first course in vector calculus; however, the topics are developed quickly to an interesting leve and for further reading there is a plethora of accessible-looking references. This book woul be unsuitable as a course textbook as it inevitably gives less space to each individual topi in order to cover more areas, but would make stimulating extra reading for the intereste applied mathematician or physicist.

Alfred Tarski – Life and Logic

By **A.B. Feferman and S. Feferman,** Reviewed by **Jacob Shepher**

Cambridge University Press, 2004.
ISBN 0-521-80240-7.

His teachers demanded rigour. Tarski delivered, becoming a brilliant, prolific logician, an yet more demanding of his own students. His work in formalising truth in model theor and entertaining problems like the volume doubling Tarski-Banach paradox are periodicall and clearly explained through the book. Despite these interludes and despite accounts o womanising, drug-use, (justified) arrogance and ambition, this biography is often circuitou and slow-reading. It's fantastically researched and comprehensive: the Fefermans ruthlessl detail and explore the influences in his life, leaving the reader wiser not just about Tarski but also relevant history, politics and his colleagues. For someone interested in the history o Tarski or logic, it's an essential book; for others, it's a formidable introduction to an equall formidable mathematician.

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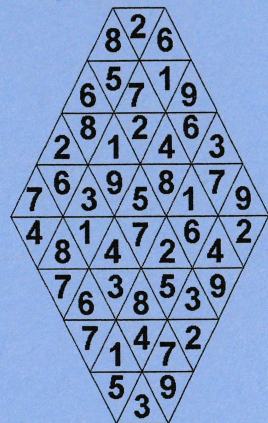
Solutions to the Problems Drive

David Chow and Vicki Wright

1. The slide rule.
2. 47160.
3. (a) ii.
   (b) viii.
   (c) xi.
   (d) iv.
   (e) x.
   (f) vi.
   (g) vii.
   (h) i.
   (i) xii.
   (j) iii.
   (k) ix.
   (l) v.
4. $\theta^2 - \sin^2\theta$.
5. The solution is:

■	■	^{1}1	^{2}2	^{3}9	^{4}6
■	■	■	^{5}9	9	7
^{6}4	^{7}2	■	■	8	■
^{8}1	0	■	^{9}1	6	■
^{10}8	2	^{11}3	3	6	■
^{12}1	5	0	■	■	^{13}3

6. (a) 2. (prime numbers)
   (b) 455. (multiples of 5)
   (c) 9450. ($\pi^{2n}/\zeta(2n)$)
   (d) 1. (abelian groups of order n)
   (e) 316. (5 less than permutations of $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, ...)
7. One possible solution is shown here:



8. Bob from Trinity owns a copy of *Eureka* 49.
9. $A = 1$, $B = 9$, $C = 6$, $D = 8$, $E = 3$.
10. $y \approx \pm 5, \pm 9, \pm 13, \ldots$ with corresponding $x \approx \log|y| \approx 2, 2, 3, \ldots$.
11. $\frac{1}{2}\sqrt{3}(x^2 - \frac{1}{3})$. $\pm x + \sqrt{\frac{1}{3}}$ will do.
12. An example is $S \cap (X \cup Z) = (S \cap X) \cup (S \cap Z)$.