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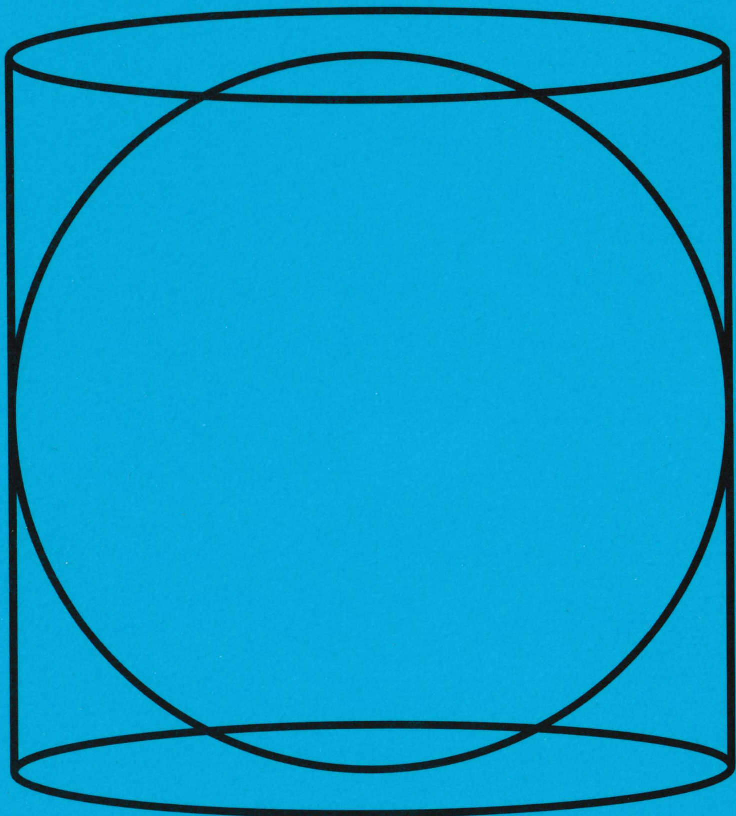
Published by The Archimedeans, the mathematics student society of the University of Cambridge

Thanks to the Betty & Gordon Moore Library, Cambridge

# Eureka



The Journal of the Archimedeans

Number 56 March 2004

## Eureka

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## Committee of the Archimedeans, 2003–2004

# EUREKA

Editor: Vicky Neale
Number 56 — Assistant Editor: Vicky Neale — March 2004
Business Manager: Stephen Burgess

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# Editorial

This is the point at which, it seems, the Editor of *Eureka* has traditionally had an opportunity to justify the existence of a student journal for recreational mathematics. Personally, I consider the existence of the journal to be a justification in itself; despite a fairly large gap since *Eureka* 55 was published, I have been very pleasantly surprised to discover how forthcoming contributions have been. In October 2003, when I became Editor, I was handed some of the material that appears on later pages, including a letter from an individual who has been reading *Eureka* for longer than I have been alive, but who still reads it as avidly as ever (indeed, it is to him that I send my thanks for some of the errata to *Eureka* 55 noted below). In the following pages, you will discover the usual assortment of items, from articles on serious subjects to pieces on less-serious subjects, and a lot in between, but at no stage have I found myself lacking people willing to contribute, and I consider this to be a very healthy situation. That said, please do not rest on your laurels; I have no doubt that whoever edits the next issue of *Eureka*, they will be similarly delighted if they are greeted by a pile of contributions and letters from appreciative readers. It only remains for me to add that I hope very much that I too shall be reading *Eureka* seventy years after I matriculated, and to hope that you enjoy what follows!

Trinity College, Cambridge
March 2004

**Acknowledgements**
As is traditional, there are numerous people whom I should like to thank, for *Eureka* 56 would not have appeared in this form and at this time without them. I am very grateful to all those who have proof-read *Eureka* for me, including Tim Austin (although naturally any remaining typographical errors are mine). The current Committee has been very helpful in organising the publication of *Eureka*; the large gap since it was last published has not made this task any easier. Huge thanks are due to Joseph Myers, who has been a constant source of helpful advice; his knowledge of copyright law and typesetting is considerably greater than mine, and his patience in explaining them has been invaluable. Although his article contains some rather intricate diagrams, he did also typeset them, for which I am very grateful, and his meticulous proof-reading has removed a significant number of errors. I must also thank Jordan Skittrall; whilst he talked me into editing *Eureka* in the first place (for which he will be forgiven once it is published!), he has been enormously helpful with technical and stylistic issues, including finding numerous mistakes in earlier drafts, and is to be thanked both for the design of the front cover (including the logo of the Archimedeans) and for moral support at frequent intervals. Lastly, and possibly most importantly, I must thank the contributors, all of whom have been immensely helpful in sending me material promptly, and who in some cases have tolerated a steady stream of questions regarding the typesetting of their articles.

# The Archimedeans, 2002–2004

## David Chow (Secretary 2003–2004)

The Archimedeans have had one of their most successful years in recent times. The very fact that you are reading this report of the Society's exploits highlights perhaps the greatest success, namely the publication of *Eureka* 56, only the second in eight years of an "annual" publication. We hope that this will mark the return of regular publication of the journal, but only time will tell.

Going back to 2002, the annual punt trip took place after the exams without too many mishaps and the garden party, held in the gardens of Selwyn, was well-attended with some entertainment being provided by a game of Twister. The following year saw interesting talks by Dr Sarah Waters (Nottingham), Professor John Greenlees (Sheffield), Professor Mark Jerrum (Edinburgh), Professor Hugh Osborn (Cambridge) and Professor Constantine Dafermos (Brown). In February, we had visitors from Oxford for the Problems Drive, this year unexpectedly won by a pair with a score of around 1.8 out of 12, even though they had not intended to take part.

At the AGM in March 2003, far shorter than the previous year's marathon effort, a new and vibrant Committee was chosen. The Easter term began with a talk by Dr Clifford Cocks (of GCHQ), which was spectacularly well-attended. After the exams came the garden party, held this year at Churchill, and the punt trip, which was almost held without punts! The Constitution of the Society was redrafted and approved at an EGM in October; this included a reduction of the size of the Committee, as you may have noticed, and the abolition of posts such as the "Procrastinator" (after many years of good service). A few teething problems inevitably ensued, including the fact that Agents of the Society were unable to resign, forcing the Committee to sack an Agent several weeks later! Professor Herbert Huppert (Cambridge), Dr Keith Crank (National Science Foundation) and Professor Tony Sudbery (York) gave fascinating and well-attended talks in the Michaelmas term, this year rounded off by a Christmas party and Puzzle Hunt.

Although the Play Reading Ideal is currently dormant, the Puzzles and Games Ring has been revived after a long absence and now meets regularly. After the Copyright Working Group had several meetings to discuss copyright issues affecting the Archimedeans, there were suggestions that a new 'armchair lawyers' subgroup might be popular with pedantic mathematicians. The famous dihebdominal newsletter has also returned after a break of several years, now appearing fortnightly online to keep members abreast of the Society's hectic calendar.

To conclude, I should now like to restart an old tradition, for various reasons not seen in *Eureka* for ten years, and write "the Archimedeans have had a very successful year".

# Errata to *Eureka* 55

- The second Problems Drive published in *Eureka* 55, by Paul Bolchover and Sean Blanchflower, was incorrectly labelled as the 1997 Problems Drive. It was, in fact, the 1998 Problems Drive.
- On page 55, the answer to Question 2 (l) should be xii, not xi.
- On page 56, the answer to Question 8 should be $\frac{\sqrt{6}}{2} - 1$, not $\frac{(\sqrt{6}-1)}{4}$.

*With thanks to A. Robert Pargeter, Joseph Myers and Sean Blanchflower*

# $p$, not $p$ or neither?

George Raptis

To all those who have not come across logical systems different from classical logic[1] before, the question above will probably seem absurd or meaningless. In this short article we aim to introduce the reader to these alternative non-classical logics and hopefully motivate him to investigate further the subject by himself.

It seems that when one tries to talk about non-classical logics, one often finds oneself "obliged" to explain the motives to introduce such ideas. In a sense, this is reasonable if we think of the long tradition of classical logic, its influence on mathematics, philosophy and science and, of course, its success in mathematics (e.g. completeness[2]) and science (e.g. classical physics). These make a revision of classical logic difficult since it demands more fundamental changes in our conceptual scheme than, for example, a revision of some scientific theory. Nevertheless, after seeing the emergence and development of some modern theories in philosophy and physics (e.g. quantum mechanics, some new views on determinism and probability, intuitionism etc.), the subject seems to be more naturally motivated or, better, it seems less absurd.

Usually, the underlying motives for non-classical logic are of a philosophical nature and hence they provide us with an endless source of discussion. Informally, we can say that non-classical logical systems were motivated by the consideration that classical logic seemed too permissive, in the sense that (classical) implication admits too many statements as true (especially, the "paradoxical" property of classical implication: $p \Rightarrow q$ is true for all $q$ if $p$ is false, which permits some arguments that seem absurd or irrelevant to be nevertheless true), or too narrow, in the sense that the assumption of classical logic that propositions are either true or false (dichotomy) is too restrictive. This resulted in the creation of intuitionistic calculi, modal logic, many-valued logics, fuzzy logic and other non-classical logics that differ from classical logic in different ways. To those logicians who find classical logic satisfactory, the arguments supporting non-classical systems sometimes seem to rely too much on epistemological (or metaphysical) issues.

Needless to mention (or if nothing else), non-classical logics should be worth studying for (mathematical) logic's sake and we can always consider this as an act of "mathematical experimentation", which may turn up to be fruitful in ways we cannot yet know.

A popular class of non-classical logics is modal logic. In modal logic we want our reasoning to involve expressions such as "it is possible that" or "it is necessary that", together with all the usual connectives of classical (propositional) logic. So we introduce a (modal) operator $\Box$, such that $\Box p$ stands for "it is necessary that $p$". After that, another operator $\Diamond$ can be defined in terms of $\Box$ by $\Diamond p = \neg\Box\neg p$ (meaning "it is possible that"), where $\neg$ is the usual propositional connective for negation. At this point, it is not clear what the truth value (which is bound to be either true or false) of $\Box p$ should be when $p$ is given to be true, so we cannot define validity in modal logic using truth tables as we did for classical propositional logic. This is usually expressed by saying that $\Box$ is not a truth-functional operator. Nevertheless, we can introduce semantics in modal logic by the idea of "possible worlds" (possible world semantics). According to this, we talk about the validity of a proposition $p$ in a possible

[1] By classical logic we mean the usual 2-valued propositional and predicate calculus as taught in a first logic course.

[2] Not all non-classical systems are complete and, of course, one can easily invent a system which is not complete. Notwithstanding how wonderful a theorem of completeness may be, at the same time it is very interesting to understand why such a theorem cannot exist for a logic.

world. A proposition $p$ may have different truth values in different worlds. More specifically, a valuation function always gives a truth value to a proposition $p$ in a world $w$ in $W$ (= all possible worlds), i.e. write $\mathrm{v}(p, w)$. Then we ask for the following to hold:

(i) $\mathrm{v}(p, w) = 1$ iff $\mathrm{v}(\neg p, w) = 0$,

(ii) $\mathrm{v}(p \Rightarrow q, w) = 1$ iff $\mathrm{v}(p, w) = 0$ or $\mathrm{v}(q, w) = 1$ and

(iii) $\mathrm{v}(\Box p, w) = 1$ iff $\mathrm{v}(p, w') = 1$ for all $w'$ in $W$.

Now, given the values of a set of primitive propositions $P$ at all possible worlds, the truth values of the propositions in the language generated by $P$ can be found. We are now ready to define validity in this system. A set of propositions $S$ semantically entails a proposition $p$ in a given set $W$ of possible worlds if and only if every valuation of the primitive propositions that assigns to the propositions in $S$ the value 1 (true) in a world $w$ in $W$ also assigns to $p$ the value 1 in the same world. We say that the argument with premises the propositions in $S$ and conclusion $p$ is valid if and only if $S$ semantically entails $p$ for every non-empty set $W$ of possible worlds. The next task is to introduce a notion of proof that creates a system that is sound, i.e. all provable statements are valid, and, if possible, also complete, i.e. a proposition is valid if and only if there is a proof for it. This is done (as in classical logic) by taking some axioms and inference rules. Let us take the following axioms:

(i) $\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$,

(ii) $\Box p \Rightarrow p$ and

(iii) $\Diamond p \Rightarrow \Box\Diamond p$

together with the inference rule:

(I) from $p$, infer $\Box p$,

and the usual axioms and inference rules of propositional logic. This system (usually called S5) turns out to be complete.

There are also other systems that differ from this in either the choice of the axioms or the choice and action of the (modal) operators, which are usually still called modal logical systems. To make this clearer, we mention here that, in modal logic, we also have a notion of accessibility, which formally is a binary relation $R$ on $W$, so that $sRt$ expresses the fact that $t$ is accessible from $s$. Then the truth value of $\Box p$ at a world $w$ depends only on the truth values of $p$ at the worlds accessible from $w$ and the dependence is in the obvious way. In the system S5, the relation $R$ is a trivial one, i.e. everything is accessible from everything else. Different binary relations $R$ provide us with different interpretations for the language, and special properties that $R$ might have (e.g. transitivity) are reflected onto the interpretation of the modal operators. That is why the term modal logic stands for a class of logical systems rather than a specific one. Maybe the most important thing concerning modal logic that the reader should keep in mind and think about is the idea of possible world semantics—the idea of frame-dependent truth values. This idea is also expressed by saying that in modal logic we distinguish between two kinds of truth: necessary truth (when $\mathrm{v}(\Box p, w) = 1$) and contingent truth (this applies to propositions that though true in some worlds can be false in others). The reader may wish to compare this with the possible-worlds interpretation of quantum mechanics.

Another class of logical systems (that will concern us in the rest of the article) is that of many-valued logic. Many-valued logic was created in the 1920s by the Polish logician and philosopher J. Łukasiewicz. The reader who is interested in the philosophical motives for many-valued logic should look for Łukasiewicz's own philosophical ideas.

Many-valued logic is different from classical logic in that it does not restrict the number of truth values to two. Systems of many-valued logic can have any finite (or even countably infinite) number of truth values.[3] The problem of interpreting the truth degrees is mainly philosophical and there is not a generally accepted interpretation. In general, it seems better not to try to agree on an interpretation but to let this depend on the field in which one applies the logic each time.

From an epistemological viewpoint, there are arguments in favour of many-valued logics due mainly to quantum mechanics. The reader is probably familiar with some of the difficulties arising in quantum mechanics when we try to describe experimental results or understand quantum mechanical notions in terms of classical logic. It has been suggested by Reichenbach that if we interpret quantum mechanics in terms of many-valued logic, it is possible that we could get rid of the conceptual problems of quantum mechanics. In a philosophical perspective, this is very important: if it is possible, it calls into question the "dogma" of the universality of classical logic and the claim of it being independent of any epistemological or metaphysical considerations.

The standard way to introduce a many-valued logical system resembles what we did for classical formal logic. We shall see how this can be done at the propositional level only. We fix a set $W$ whose values are the truth degrees and a set of designated truth degrees, which is a subset of the set of truth degrees and whose elements act as substitutes for the traditional truth value of "true".

Semantics in many-valued logics are most times conveniently defined by truth tables as in classical logic. Truth tables will show how the valuation function "interprets" the primitive propositional connectives. The size of the truth tables depends on the number of the truth degrees in the logic. We define the notion of semantic entailment similarly to the classical logic (where now a proposition is true (valid) if and only if all valuation functions give it a designated truth degree).

The next step, as in classical logic, is to introduce a notion of proof in the logic. We can do that by fixing a set of axioms together with a set of inference rules. Next, we define proof and syntactic entailment as usual. In the following we shall see how we can apply these to define Łukasiewicz's systems.

It is reasonable to start with 3 values: 0, $\frac{1}{2}$, 1. If we reserve 1 for true and 0 for false, how can we interpret $\frac{1}{2}$? If we interpret $\frac{1}{2}$ as the truth degree of meaningless (ill-stated) statements, then we do not gain much by introducing this new system, since in classical logic we restrict attention to well-formed statements (formulae) anyway. So we have to understand here that any attempt to build a non-classical system has to be committed first to a conceptual change about validity or provability or something else equally fundamental. Also, we should mention here that since we use common English when describing many-valued logical systems (and when stating metatheorems about them), we do not change the logic used in the metalanguage.

A Łukasiewicz system is a many-valued system $L_m$, for $m$ a positive integer, that has a truth degree set of $m$ elements, which we usually write as $W_m = \left\{\frac{k}{m-1} : 0 \leq k \leq m-1\right\}$. In this system, the degree 1 is the only designated truth degree.

[3]There are similar many-valued logical systems where uncountably many truth degrees are allowed. In fuzzy logic, truth degrees range over the interval $\{x : 0 \leq x \leq 1\}$, but we keep it conceptually distinct from many-valued logics.

The main propositional connectives, from which the language is built (as in classical propositional calculus), are an implication $\rightarrow$ ("implies") and a negation $\neg$ ("not"). So the Łukasiewicz system has the same well-formed formulae (wff) as the classical system. Now, we can define how the truth function (valuation) v applies to these propositional connectives by the formulae $\mathrm{v}(u \rightarrow v) = \min\{1, 1 - \mathrm{v}(u) + \mathrm{v}(v)\}$ and $\mathrm{v}(\neg u) = 1 - \mathrm{v}(u)$. Then the value of any wff in the language can be determined given the truth values of the primitive propositions.

We see from these definitions that if we restrict our attention to only the two truth values of 0 and 1, we obtain the truth tables of classical connectives in two-valued logic. This shows that $L_m$ "generalizes" classical propositional calculus (note that $L_2$ is the system of classical propositional logic) and that the set of tautologies of $L_m$ is a proper ($(p \rightarrow \neg p) \rightarrow \neg p$ is not always true in $L_m$) subset of the set of tautologies of classical logic. Hence this system is less permissive (has fewer valid propositions—tautologies) than classical propositional logic.

All other propositional connectives can be defined in terms of $\rightarrow$ and $\neg$, for example, we can define the disjunction $\vee$ analogously to classical logic as $p \vee_1 q \equiv \neg p \rightarrow q$. However, this definition of disjunction is not satisfactory because it does not preserve the desirable properties of $\vee$ in $L_m$. In particular, $p \vee_1 p$ does not always take the same truth value as $p$. Moreover, in order to make the definition of disjunction intuitively more sound, we should like the truth value of the disjunction of two propositions to be the maximum of the truth values of these propositions. To this end, we define $p \vee_2 q \equiv ((p \rightarrow q) \rightarrow q)$, which satisfies this requirement and has the properties we want. Thus in many-valued logic we have two kinds of disjunction (sometimes called weak and strong respectively) and the same, of course, applies to the conjunction $\wedge$.

A characteristic property of many-valued logic is that the statement "$p$ is true if and only if $\neg p$ is false" does not hold in general. However, the negation of a designated (true in many-valued logic) proposition is undesignated (i.e. never designated). In order to define an operator that behaves as negation does in classical logic, Rosser and Turquette introduced the J-functions. The interested reader may wish to consult their classic but rather old book on many-valued logics.

There are many different axiomatizations for the Łukasiewicz system that have been proved to be complete.[4] Some of these systems have elegant axiomatizations and hard proofs for completeness, while others are not elegant but they faciliate the proof of completeness. Writing down proofs in the Łukasiewicz system (with some axiomatization) is, in general, harder than in classical logic. One reason for this is that the deduction theorem is no longer valid in many-valued logic (why?).

Finally, we could remark that non-classical logic is a complex system of many different logical systems and, perhaps, it is more interesting to see how they are connected to each other (and to classical logic) than to argue which one is best or the right one. Also we should say that here we restricted attention to the propositional calculus of the logical systems discussed; predicate calculi can, of course, be defined for non-classical logics as well, but to do this we should need another article.

[4]For example, the axiomatizations suggested by Wajsberg or the one proposed by Rosser and Turquette (using J-functions).

# Some Notes on the Theory of Cycling

David Loeffler

## Introduction

All Cambridge students and ex-students will be aware that the bicycle is the fastest and most efficient means of transport in the congested environment of central Cambridge. However, for some inexplicable reason, not all of the undergraduate population acknowledges this to the extent of actually owning a bicycle.

So it was that last term, a friend and I were discussing the possibility of attending some early-morning lectures at the CMS *[Ed: Centre for Mathematical Sciences]*. My friend is one of these ignorant hordes who do not own bicycles, and we spent some while discussing whether or not it was possible to share this one bicycle between us efficiently to minimise our travel time, thus enabling us to get up as late as possible. This article accordingly contains exactly one theorem, which gives a general construction for the most efficient method to do so.

## A simple case

Let us consider the simplest non-trivial case of this problem, which was that which initially concerned my friend and me. This is that of two individuals, sharing one bicycle between them.

(Hereafter it will be assumed that all people cycle at the same constant speed, and walk at the same speed, and that the speed of cycling is greater than that of walking. It will also be assumed that bicycles do not move unless they are being ridden by somebody, and that nobody except myself and my friend will ride these bicycles.)[1]

The solution that we reached was as follows: my friend and I should depart at the same time from college, him on foot and myself cycling. When I reached the midpoint of the journey from there to the CMS, I should dismount and abandon the bicycle, continuing my journey on foot.

My friend would reach the midpoint of the journey some time after I had left it, and would then take my abandoned bicycle and ride it to the CMS. A moment's thought shows that we should arrive at the CMS at the same time.

Now, I claim that this is the optimum strategy—that no other method of sharing out the bicycle would allow both of us to complete our journey in less time.

The proof is very simple if one assumes a seemingly trivial hypothesis: that the optimum strategy does not involve cycling backwards at any point.

With this in mind, we see that if I rode the bicycle for $x$ times the total distance (which we may normalise to 1), I should take time $x/v_{cyc} + (1-x)/v_{walk}$. My friend would then take at least $(1-x)/v_{cyc} + x/v_{walk}$. Thus the mean of our times would be $\frac{1}{2}(1/v_{walk} + 1/v_{cyc})$, independent of $x$. Thus the maximum of our two times must be at least this, with equality attained only if both times are equal—that is, if $x = \frac{1}{2}$, which is the strategy outlined above.

## Backwardness

It remains to prove the hypothesis that cycling backwards does not improve the situation. This is intuitively obvious, but requires careful proof—one is reminded of $\mathbb{QARCH}$ problem 75

[1] We are also making some other assumptions, including the assumption that bikes may be instantaneously and perfectly securely locked and unlocked at any desired location!

from *Eureka* 55, which has the same superficial obviousness.

Consider any particular stretch of the journey: we shall assume this is short enough that no bicycle swaps occur along this stretch. Let us consider the total time spent traversing this stretch.

Evidently both myself and my friend must traverse the stretch. We desire to show that the total time (in terms of person-hours or whatever) taken for us to do so is minimised when exactly one of us is cycling.

However, this is obvious: there being only one bicycle, if neither of us covers this segment of the journey on foot, one of us must have left the bicycle at the nearer (to the start) endpoint of this segment, in order that the other might use it; this person must now traverse the interval on foot, and the aggregate journey time is thus more than the solution above.

### Larger populations

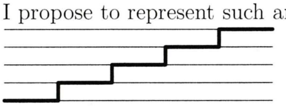
With the above in hand, it is natural to consider the following more general problem: we now have $n$ people sharing $m$ bicycles between them, where $m < n$ (otherwise the problem is evidently slightly trivial).

It is easy to see that the logic above implies that the total time taken in person-hours must be at least the sum of $m$ times the time to cycle this distance and $n - m$ times the time to walk it. However, the main focus of this article will be to show that we can achieve this total time in such a way that all the people concerned arrive simultaneously, thus minimising the time taken until all have arrived.

We shall study two easy cases first.

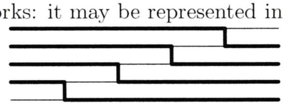
**$m = 1$** In this case, there exists an attractive solution: one individual starts cycling, and the other $n - 1$ start walking. The first individual stops when he has covered $1/n$ of the distance, and leaves the bicycle, continuing on foot to the destination. The second individual will, meanwhile, have reached the bicycle at $x = \frac{1}{n}$, and will ride on to $x = \frac{2}{n}$, before dismounting and walking the rest of the way.

I propose to represent such an arrangement by a graph of this form:



Here the $x$-axis represents distance, and the various horizontal lines are the paths of the people (5 of them in the above diagram). The heavy line represents the path of the bicycle.

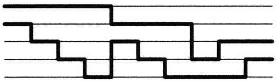
**$m = n - 1$** In this case, an arrangement that is almost precisely the reverse of the above works: it may be represented in the same way by



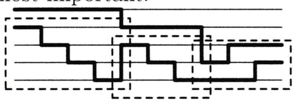
**The general case** In order to show that we can share out bicycles equally for entirely general $m$ and $n$, we shall build up from the two cases above.

Let us suppose we want to do the case $n = 5$, $m = 2$, using the results given above. Then we can imagine dispatching one cyclist to cycle $\frac{2}{5}$ of the distance, before dismounting and leaving his bicycle behind and walking the rest of the way. Meanwhile, we shall persuade the remaining 4 people to share out the one bicycle they have left in the manner described above as if they were all attempting to travel to the point $\frac{2}{5}$ along the way.

They will all arrive at this point simultaneously, and the remaining bicycle will already be there. Then what do they do? Of course, they solve the problem $n = 4$, $m = 2$ over the remaining distance. This is of course solvable by splitting them into two groups and solving $n = 2$, $m = 1$ for these groups, but we could also repeat the process again to illustrate the method. Then we obtain the following rather charming graph:



I have adjusted this artificially so that the paths of the bicycles do not appear to "cross over one another"; although this is solely a phenomenon of how the system is being represented, it is nonetheless somewhat confusing. However, the modular structure of this graph is what is most important:



Each boxed section is an example of one of the two graphs above. It is worth noting that both of the standard graphs may be built up recursively in this manner.

**Theorem 1** *For any $n$ and $0 \leq m \leq n$, there exists an arrangement whereby every passenger uses a bicycle for exactly $\frac{m}{n}$ of the distance.*

**Proof** Evidently for $n = 1$ the result is trivial. Hence let us assume that the problem is solved for all $n' < n$ and all corresponding $m$.

If $m = 0$ the problem is not difficult: everyone walks. Likewise, if $m = n$ everybody cycles.

Otherwise, let us dispatch one passenger to cover the first $\frac{m}{n}$ of the distance on one of the bicycles and leave this bicycle at that point, and walk thereafter.

Then let us marshal the remaining cyclists to solve the problem of $n-1$ passengers on $m-1$ bicycles over the distance from the start to the point $\frac{m}{n}$ of the distance, which we may do by the induction hypothesis, and then solve the problem of $n-1$ passengers on $m$ bicycles over the remaining distance, which makes sense since we are dealing with the case $m \leq n-1$.

It remains to check that each passenger has covered exactly $\frac{m}{n}$ of the distance cycling. However, this is evidently true for the first passenger; and for the others, they will cover $\frac{m-1}{n-1}$ of the first subsection of the distance, and $\frac{m}{n-1}$ of the second, by bicycle, according to the induction hypothesis. That is, their total cycling distance is

$$\begin{aligned} & \frac{m-1}{n-1} \cdot \frac{m}{n} + \frac{m}{n-1} \cdot \frac{n-m}{n} \\ = & \frac{m}{n(n-1)} (m - 1 + n - m) = \frac{m}{n} \end{aligned}$$

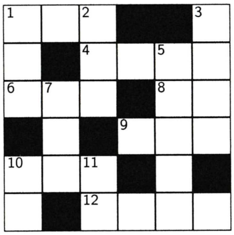
as required. $\square$

# Problems Drive 2003

Toby Kenney and Paul Russell

**Question 1.** What is the smallest positive integer that *cannot* be made with the four elementary arithmetic operations using the numbers 1, 5, 6 and 7 precisely once each?
*[Note that concatenation of digits is not permitted.]*

**Question 2.** Solve the following cross-number. No answer begins with a zero.




ACROSS

1. 2 × a prime.
4. 13 × a $4^{th}$ power.
6. A sum of 2 cubes.
8. A prime.
9. A $6^{th}$ power.
10. A prime power, but neither a prime nor the square of a prime.
12. The square of a prime.

DOWN

1. 4 × (10 down).
2. (1 across) + (9 across); a prime.
3. All of its prime factors are palindromic (1-digit primes count as palindromic).
5. 2 × a $4^{th}$ power.
7. A multiple of (11 down).
10. A prime.
11. A prime.

**Question 3.** Below are given anagrams of the names of twelve mathematicians of note and twelve dates; match up the mathematicians with their dates of birth.

| | |
|---|---|
| (I) BLANK AREA | (i) 276 B.C. |
| (II) FAR DEEPER MERIT | (ii) c.200 A.D. |
| (III) FETCH BANANAS | (iii) $19^{th}$ February 1473 |
| (IV) HANDS OUT PI | (iv) $17^{th}$ August 1601 |
| (V) HEARD EXALTED RECKONING | (v) $25^{th}$ December 1642 |
| (VI) I GOAD HIM WELL | (vi) $21^{th}$ August 1789 |
| (VII) ORNATE THESES | (vii) $30^{th}$ March 1892 |
| (VIII) SMOOTH GREY WIT | (viii) $17^{th}$ June 1903 |
| (IX) SPECIOUS LUNAR CONIC | (ix) $28^{th}$ March 1928 |
| (X) UNUSUAL GUY IS CHAOTIC | (x) $19^{th}$ August 1939 |
| (XI) WANT A COSINE | (xi) $11^{th}$ April 1953 |
| (XII) WIELD ANSWER | (xii) $20^{th}$ November 1963 |

**Question 4.** Find a positive integer $n > 1$ such that the arithmetic mean of the squares of the first $n$ positive integers is itself a perfect square.

**Question 5.** Codge and Hollier play the following game on a $5 \times 3$ chessboard:
Initially, a chess knight is placed on the square marked X. At each time interval, an unbiased coin is tossed; if it comes up heads, Codge may move the knight, while if it comes up tails then Hollier may move the knight. Codge wins if the knight reaches the square marked C, while Hollier wins if the knight reaches the square marked H.
Assuming best play by both sides, what is the probability that Codge wins?
*[The move of a knight is two squares either horizontally or vertically followed by one square in a direction orthogonal to that of the first part of the move.]*

| | | |
|---|---|---|
| | | C |
| | | X |
| | | |
| | | |
| | | H |

**Question 6.** One year in Tripos, the top five places are taken by students with names David, David, James, Fred and Tom. Three are from Trinity and two are from Caius. Two are number theorists, two are statisticians and one is a logician. Despite the names, two are Hungarian, two are Russian and one is British. It is given that:

At least one of the Davids is from Trinity.
If there is a number theorist at Caius then he is British.
All Russian statisticians are at Trinity, and there is at least one.
The person who came top is not Russian.
If the person who came top is at Caius then the person who came $4^{\text{th}}$ is at Trinity.
There are at least one number theorist and at least one statistician at Trinity.
Neither James nor Tom is at Caius.
If Fred is at Caius then all number theorists are at Trinity.
The British student is called either David or Tom.
If James is Russian then he is a statistician.
The person who came $3^{\text{rd}}$ is a number theorist.
If the person who came $4^{\text{th}}$ is Hungarian then so is the person who came $5^{\text{th}}$.
Any David who is a number theorist is Hungarian.
One of the students who came $2^{\text{nd}}$ and $4^{\text{th}}$ is at Caius (but not both).
Any David who is a statistician is British.
If the person placed $3^{\text{rd}}$ is at Trinity then the person placed $4^{\text{th}}$ is at Caius.
If all Hungarians are at Trinity then all number theorists are at Caius.

Where did Fred come?

**Question 7.** Find the next term in each of the following sequences:

(i) 1, 9, 36, 100, 225, 441, 784, 1296, ???
(ii) 1, 4, 27, 256, 3125, 46656, 823543, 16777216, ???
(iii) 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ???
(iv) 3, 7, 14, 23, 36, 49, 66, 83, 104, 129, 152, ???
(v) 0, 1, −3, 24, 8, 133, 97, 440, 376, 1105, 1005, ???
(vi) 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, ???

**Question 8.** Arrange in increasing order of magnitude the collection of numbers

$$\{x^y : x \in X, y \in Y\} \cup \{y^x : x \in X, y \in Y\}$$

where

$$X = \{2\sqrt{2}, e, \frac{11}{4}\} \quad \text{and} \quad Y = \{\pi, \sqrt{10}, 3\}$$

**Question 9.** List all positive integers $n < 1000$ such that the sum of the digits of $n$ is equal to the sum of the digits of $2002n$.

**Question 10.** If MATHS=(FUN)$^2$ and MAN is prime, which digit is represented by each letter in the above sum?
*[Each letter represents a different digit chosen from 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9; there are no leading zeros.]*

**Question 11.** Let $ABCD$ be a cyclic quadrilateral whose circumcircle has centre $O$ and radius 1. Let $AC$ and $BD$ meet at $X$. Suppose that $AC = \sqrt{3}$, $BD = \sqrt{2}$ and $OX = \frac{\sqrt{3}}{2}$. What are the possible values for the area of the quadrilateral $ABCD$?

**Question 12.** Béla is thinking of two distinct positive integers not greater than 10 and makes the following statements, of which precisely four are false:

(i) The smaller number is a square.
(ii) The larger number is a cube.
(iii) One of the numbers is a prime.
(iv) Neither of the numbers is 9.
(v) The smaller number is the number of true statements made before this one.
(vi) The number of true statements made before this one is less than the larger number.
(vii) The number of true statements made before this one is the smaller number.
(viii) The smaller number is not prime.
(ix) The larger number is a sum of two distinct squares.
(x) The smaller number is a sum of two triangular numbers.
(xi) The larger number is the number of true statements made before this one.

Identify Béla's numbers.

# Partial Orders and Topologies on Finite Sets

Demetres Christofides and George Raptis

A quasi-ordered set $(X, \prec)$ is a set $X$, together with a binary relation $\prec$ which is reflexive and transitive. The relation $\sim$ on $X$ defined by $x \sim y$ if $x \prec y$ and $y \prec x$ is an equivalence relation on $X$. We can define a relation $\leq$ on $X/\!\sim\, = \{[x] : x \in X\}$ by $[x] \leq [y]$ if $x \prec y$. This is a well-defined quasi-order which is also antisymmetric. A quasi-ordered set $X$ with a relation $\leq$ which is also antisymmetric is called a partially ordered set (poset). For $x$, $y \in X$ we write $x < y$ meaning $x \leq y$ and $x \neq y$. We say that $x$, $y \in X$ are incomparable if neither $x \leq y$ nor $y \leq x$.

The number of quasi-orders on a set of $n$ elements is denoted by $Q_n$. The number of partial orders on a set of $n$ elements is denoted by $P_n$. There are no explicit formulae for $P_n$ and $Q_n$ and the problem of finding such formulae is open and possibly very hard (see the table below).

| $n$ | $P_n$ | $Q_n$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 3 | 4 |
| 3 | 19 | 29 |
| 4 | 219 | 355 |
| 5 | 4231 | 6942 |
| 6 | 130023 | 209527 |

We shall now mention some of the existing results concerning those sequences.

It is easy to see that there are exactly $3^{\binom{n}{2}}$ reflexive and antisymmetric relations on a set $X$ of $n$ elements. (For every two distinct $x$, $y \in X$, either $x \leq y$, or $y \leq x$ or $x$ and $y$ are incomparable.) In particular,

$$P_n \leq 3^{\binom{n}{2}}.$$

Partitioning $X$ into $k$ non-empty subsets $X_1, \ldots, X_k$ and partially ordering the set of these subsets, we get a quasi-order on $X$. Clearly we can get every quasi-order on $X$ uniquely in this way. If we denote by $S(n, k)$ the number of ways to partition a set of $n$ elements into $k$ non-empty subsets (this is known as the Stirling number of the second kind), we get that

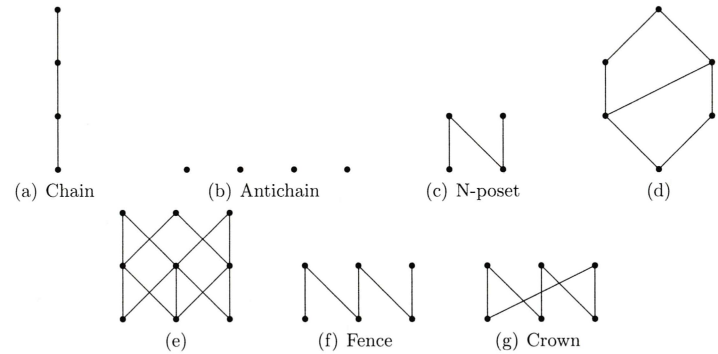
$$Q_n = \sum_{k=1}^{n} S(n, k) P_k.$$

It is also known that $\lim\limits_{n \to \infty} \frac{P_n}{Q_n} = 1$. For this, the reader may consult [3].

Let $(X, \leq)$ be a finite poset. We say that an element $x \in X$ covers $y \in X$ if $y < x$ and there does not exist $z \in X$ with $y < z$ and $z < x$. We can represent $X$ with a diagram, by representing each element $x \in X$ by a distinct point, so that

(i) Whenever $x < y$, the point representing $y$ is higher than the point representing $x$. (Imagine this on the plane.)

(ii) Whenever $y$ covers $x$, the points representing the two elements are joined by a straight line segment.



(a) Chain (b) Antichain (c) N-poset (d)

(e) (f) Fence (g) Crown

This diagram is called the Hasse diagram of the poset $X$. (See Figures (a)–(g).)

We can view the Hasse diagram of a poset as a directed graph. (There is a directed edge from $x$ to $y$ if $y$ covers $x$.) Clearly, every such directed graph cannot have either a directed cycle (by transitivity of $\leq$) or a cycle with all but one edge directed cyclically (by construction). It turns out that these two conditions are also sufficient for a directed graph to represent a poset. (Interpreting each directed edge $\overrightarrow{xy}$ as '$y$ covers $x$'.) In particular, such a directed graph can have no triangles. Every bipartite graph on $n$ vertices, with the edges directed from one partite set to the other, satisfies the above conditions. By considering only those with $\lfloor \frac{n}{2} \rfloor$ vertices in one partite set, and $\lceil \frac{n}{2} \rceil$ in the other, we deduce that there are at least $2^{\lfloor \frac{n^2}{4} \rfloor}$ such directed graphs, and so

$$P_n \geq 2^{\lfloor \frac{n^2}{4} \rfloor}.$$

In fact, Kleitman and Rothschild in [5] proved that there is a constant $C$ such that

$$\log_2 P_n \leq \frac{n^2}{4} + C n^{\frac{3}{2}} \log_2 n$$

for all $n$. Hence they deduced that

$$\log_2 P_n \sim \frac{n^2}{4}.$$

Now we shall see how the problem of enumerating partial orders on a set $X$ of $n$ elements turns out to be equivalent to the problem of enumerating topologies on $X$ with a certain property. We give first some definitions:

A topological space $(X, \mathcal{T})$ is said to be a $T_0$-space if, for every distinct $x$, $y \in X$, there is an open set $U$ containing exactly one of them.

A topological space $(X, \mathcal{T})$ is said to be a $T_1$-space if, for every $x \in X$, $\{x\}$ is closed.

A topological space $(X, \mathcal{T})$ is said to be an $A$-space if any intersection of open sets is open. In particular any topology on a finite set is an $A$-topology.

**Lemma 1** *Let $X$ be a topological space.*

*(i) If $X$ is $T_1$, then $X$ is also $T_0$.*

*(ii) If $X$ is an $A$-space, then the only $T_1$-topology on $X$ is the discrete topology.*

*(iii) If $X$ is a finite $T_0$-space, then there is an $x \in X$ such that $\{x\}$ is closed.*

**Proof**

(i) Follows from the observation that $X - \{x\}$ is open.

(ii) Since an arbitrary union of closed sets is closed.

(iii) By induction. □

**Theorem 2** *Let $X$ be a finite set with $n$ elements. Then*

*(i) there is a 1-1 correspondence between the set of quasi-orders on $X$ and the set of topologies on $X$, and*

*(ii) there is a 1-1 correspondence between the set of partial orders on $X$ and the set of $T_0$-topologies on $X$.*

**Proof**

(i) Given a quasi-order $\prec$ on $X$, we define a topology on $X$ which has basis of open sets the sets of the form $U_x = \{y \in X : y \prec x\}$ for $x \in X$. This is indeed a basis since $U_x \cap U_y = \bigcup\{U_z : z \prec x \text{ and } z \prec y\}$. Conversely, given a topology $\mathcal{T}$ on $X$ we can define a quasi-order $\prec$ on $X$ by $x \prec y$ if and only if $V_x \subset V_y$ where $V_x = \bigcap\{U \in \mathcal{T} : x \in U\}$. It is clear that $\prec$ is a quasi-order on $X$, and the reader can check that these maps are mutual inverses.

(ii) Using the above correspondence, it suffices to show that a partial order corresponds to a $T_0$-space and vice-versa. Given a partial order $\leq$ on $X$, and distinct $x, y \in X$, then either $x$ and $y$ are incomparable, in which case $x \in U_x$ but $y \notin U_x$, or they are comparable, say (without any loss of generality) $x \leq y$, in which case $x \in U_x$ but $y \notin U_x$. Conversely, given a $T_0$-topology on $X$, and two elements $x, y \in X$ with $x \prec y$ and $y \prec x$, then $y$ belongs to every open set containing $x$ and $x$ belongs to every open set containing $y$. Therefore $x = y$, hence $\prec$ is a partial order. □

**Remarks**

(1) The above proof is not valid for infinite sets because the two maps in (i) might no longer be inverses to each other, since $V_x$ is not open in general.

(2) For an infinite set $X$, the above correspondence holds if we consider only $A$-topologies on $X$. The proof is essentially the same as above and is therefore left to the reader.

(3) In connection to this theorem, Lemma 1 says that the only poset (finite or infinite) corresponding to a $T_1$-topology is the antichain and that every finite poset has a maximal element.

(4) In [9] it is shown that on a set $X$ of $n$ elements, no topology other than the discrete has more than $\frac{3}{4}2^n$ open sets. This result is best possible, i.e. for $n \geq 2$ there is a topology on $X$ with exactly $\frac{3}{4}2^n$ open sets. In [10], Stanley proved a stronger result.

Given a relation $R$ on a set $X = \{x_1, \ldots, x_n\}$, we can define an $n \times n$ $(0,1)$-matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i R x_j, \\ 0 & \text{otherwise.} \end{cases}$$

This gives a 1-1 correspondence between relations on $X$ and $n \times n$ $(0,1)$-matrices, since $R$ is a subset of $X \times X$. In the case where $R$ is a quasi-order, the corresponding matrix $A$ satisfies:

(i) $a_{ii} = 1$ for every $i$, and

(ii) if $a_{ij} = a_{jk} = 1$, then $a_{ik} = 1$.

Conversely, we leave the reader to check that these conditions on $A$ determine uniquely a quasi-order on $X$. In the case where $R$ is a partial order, then $A$ also has to satisfy:

(iii) for all $i, j$, $a_{ij} = a_{ji} = 1$ if and only if $i = j$.

Hence, by Theorem 2, there is a 1-1 correspondence between finite topological $T_0$-spaces on $X$, and $(0,1)$-matrices satisfying conditions (i), (ii) (and condition (iii)) above. This correspondence can be described in terms of the topology directly by:

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in \overline{\{x_i\}}, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 3** *A (0,1) $n \times n$ matrix $A$ corresponds to a topological space (a quasi-order) on a set $X$ of $n$ elements, if and only if it has 1s on the main diagonal and satisfies $A^2 = A$ under Boolean operations*[1].

**Proof** Such a matrix $A$ clearly satisfies (i). If $a_{ij} = a_{jk} = 1$ for some $i, j, k$, then $a_{ik} = \sum_{j=1}^{n} a_{ij}a_{jk} = 1$, hence $A$ also satisfies (ii). Conversely if a matrix $A$ satisfies (i) and (ii), then we need to show that $a_{ik} = \sum_{j=1}^{n} a_{ij}a_{jk}$. If $a_{ik} = 0$, then we cannot have $a_{ij} = a_{jk} = 1$ for any $j$ (by condition (ii)). If $a_{ik} = 1$, then $a_{ii}a_{ik} = 1$, so $\sum_{j=1}^{n} a_{ij}a_{jk} = 1$. □

A map $f$ between two posets $X$, $X'$ is called order preserving if, for all $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$, where $\leq$ denotes the partial order in both $X$ and $X'$. A poset $X$ is said to have the fixed point property if every order preserving map $f : X \to X$ has a fixed point, i.e. there is an $x \in X$ with $f(x) = x$. The poset shown in Figure (e) has this property. The problem of characterizing the posets with the fixed point property is open. This has been answered in the positive (Knaster-Tarski theorem) in the case of complete posets[2]. (See Figure (d) for the Hasse diagram of a complete poset.)

In the case of lattices[3], the above problem has been completely solved. Tarski in [11] proved that every complete lattice[4] has the fixed point property and Davis in [2] proved that every lattice with the fixed point property is complete.

[1] i.e. $1+1 = 1+0 = 0+1 = 1 \times 1 = 1$, $0+0 = 0 \times 0 = 0 \times 1 = 1 \times 0 = 0$.

[2] A poset $X$ is complete if, for every subset $Y$ of $X$, there is an element $x \in X$ with $x \geq y$ for all $y \in Y$ and for all $z \in X$ with this property, $z \geq x$. We call $x$ the least upper bound of $Y$.

[3] A lattice $X$ is a poset with the additional property that every two distinct $x, y \in X$ have a least upper bound and a greatest lower bound, the latter defined in the obvious way. For example, the power set of a set ordered under inclusion.

[4] A lattice $X$ is complete if every subset of $X$ has a least upper bound and a greatest lower bound. In particular, every finite lattice is complete. See Figure (d) again.

It is now reasonable to ask what the relationship between the maps that preserve the order structure and the maps that preserve the corresponding topological structure is. We shall end our introduction to posets by completing the correspondence between partial orders and $T_0$-topological spaces with the following lemma:

**Lemma 4** *A map $f$ from a poset $X$ into $X$ is order preserving if and only if it is continuous with respect to the corresponding topology on $X$ (as given in the proof of Theorem 2).*

**Proof** If $f$ is order preserving, then $f^{-1}(U_x) = \bigcup\{U_z : z \leq f(x)\}$ is open. Therefore $f$ is continuous. Conversely, if $f$ is continuous and $x, y \in X$ with $x \leq y$, then $U_x \subset U_y$. Since $f$ is continuous, $f^{-1}(U_{f(y)})$ is an open set containing $y$, hence it also contains $U_y$ and so it also contains $U_x$. It follows that $f(x) \in U_{f(y)}$, i.e. $f(x) \leq f(y)$. Therefore $f$ is order preserving. □

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*Ed: I have received the following contribution from A. Robert Pargeter, who matriculated at Pembroke College in 1934, and who may reasonably be described as a founder member of the Archimedeans. In the letter accompanying the piece, he noted the recent publication of various books on the subject of zero and nothing.*

# Nothing

A. Robert Pargeter

"I think I will write an essay about nothing."

"Nonsense."

"No, nothing."

"What do you mean, 'no, nothing'?"

"Just what I say. Nothing is not nonsense."

"I didn't say it was: your essay will be nonsense."

"My essays are never nonsense."

"So you say—but what I meant, of course, is that the idea of writing an essay on nothing is nonsense."

"Why? Surely one can write an essay on anything."

"Quite; but by definition, nothing is not anything."

"Which, by virtue of an undistributed middle, proves nothing—the fact that one *can* write an essay on anything does not imply that one *cannot* write an essay on nothing."

"By the same token it does not imply that one can, since anything is not nothing. But of course the thing is, one can't say anything about nothing."

"By that you contradict yourself, and justify me. For if it is true that one cannot say anything about nothing, then that is itself a statement about nothing, while if it is false, then there is something one can say about nothing. What did you say?"

"Oh, nothing."

"Now the problem as I see it is not that there is not much to be said about nothing, but that the possibilities are too abundant. How in fact shall I begin? Perhaps an imaginary dialogue with an argumentative friend. . . ."

D.C.

# Back Issues of *Eureka*

Copies are available of some earlier issues of *Eureka*. At the time of going to press, *Eureka*s 53, 54 and 55 are £1.50 each (plus postage and packing), and the others are £1 each (plus postage and packing). There are very few copies of some issues left, so please look on the *Eureka* website (`http://www.cam.ac.uk/societies/archim/eureka/`) for details of which issues are available. For more information, please contact the Subscriptions Manager (at the address inside the front cover).

# Tiling with Regular Star Polygons

Joseph Myers

The Archimedean tilings (Figure 1) and polyhedra will be familiar to many readers. They have the property that the tiles of the tiling, or the faces of the polyhedron, are regular polygons, and that the vertices form a single orbit under the symmetries of the tiling or polyhedron. (Grünbaum and Shephard [1] use *Archimedean*, in relation to tilings, to refer to the sequence of polygons at each vertex being the same, and *uniform* to refer to the vertices forming a single orbit. These describe the same set of tilings, but the sets of $k$-uniform tilings (those with $k$ orbits of vertices) and $k$-Archimedean tilings (those with $k$ different types of vertices) differ for $k > 1$. I do not make this distinction in this article, but use the term *uniform* to avoid ambiguity. In relation to polyhedra, the distinction made between these terms is different.)

The Archimedean polyhedra were attributed to Archimedes by Pappus [3, 4], although the work of Archimedes on them has not survived; the tilings may have been named by analogy. The first surviving systematic account of either the tilings or the polyhedra seems to be that of Kepler [5, 6]. The literature on the 2-uniform and 3-uniform tilings is discussed by Grünbaum and Shephard [1]; the $k$-uniform tilings for $k \leq 6$ are presented by Galebach [15].

Given these tilings and polyhedra, for centuries people have generalised in different ways (for example, through changing the definitions of tilings and polyhedra, through changing the permissible tiles and faces, or through considering analogous concepts in higher dimensions). Some of these generalisations have yielded more aesthetically pleasing results than others. One form of generalisation, considered by Kepler, is allowing star polygons. Two different types of regular star polygons may be considered. One, the modern version, treats a star $n$-gon as a polygon with $n$ edges, which intersect each other; only the $n$ endpoints of those edges are considered as corners of the polygon, and not the points of intersection of the sides.[1] The notion of 'tilings' with such polygons is not very clear, but it has been considered thoroughly [7]; many of these tilings are not especially aesthetically pleasing because of the density of the crossing lines that make up the edges of the polygons, and a single drawing can represent multiple distinct tilings. When polyhedra with such polygons as faces are considered, the set of uniform polyhedra [8, 9, 10, 11, 12] appears; these are rather more attractive; some readers may have seen the author's models of some of these polyhedra on the Archimedeans' Societies Fair stand in 2002. Kepler considered regular polyhedra with this notion, finding the small and great stellated dodecahedra but not the great dodecahedron or great icosahedron which were later found by Poinsot [13].

The other type of regular star $n$-gon (guided more by aesthetics than by mathematical generalisation) is a nonconvex $2n$-gon with equal sides and alternating angles; *$n$ points* of angle $\alpha$ (with $0 < \alpha < (n-2)\pi/n$) and *$n$ dents* of angle $2(n-1)\pi/n - \alpha$; we denote this polygon $n_\alpha$. When considering tilings, Kepler used this notion; he drew various patches of tilings using such polygons, and mentioned various such tilings found in the course of enumerating the uniform tilings with regular convex polygons. However, he never made it clear exactly which polygons and tilings were allowed. This type of regular star polygon yields more attractive tilings than the modern more mathematical type of star polygon (which yields tilings rather too densely cluttered with lines). It would be natural mathematically to consider

[1] With this version, a polygon is considered regular if its symmetries act transitively on the pairs (vertex, edge incident with that vertex). Infinite polygons, aperiogons and zigzags, may be allowed; when they are, [7] notes that their enumeration of tilings is only conjectural.

polygons with equal sides and alternating angles, convex or nonconvex, but given the aesthetic and historical motivation we do not do so here.

Grünbaum and Shephard [2] made the first attempt at enumeration of such tilings (and so some sort of completion of Kepler's enumeration) under definite rules, attempting to find uniform or $k$-uniform tilings with regular polygons and any $n_\alpha$ star polygons. In [1] they adjusted the definitions used, so that the *vertices* of the tiling are only those points where three or more tiles meet; if a dent of a star is filled entirely by the corner of one other polygon, that is a corner of the polygons but not a vertex of the tiling. They also consider tilings that are not edge-to-edge: where the polygons involved may have different edge lengths, and some vertices are in the middle of edges. They presented drawings of uniform tilings involving star polygons, which they conjectured show all such tilings, giving as an exercise proving that their list of uniform tilings involving star polygons which are not edge-to-edge is complete, and another exercise asking whether there were any other (edge-to-edge) uniform tilings involving star polygons. Apart from this work and that of Kepler, such tilings do not seem to have been considered in the mathematical literature, although some are shown in [14].

When I attempted those exercises in 1993, it turned out, however, that those lists were not complete; there are three uniform tilings, one of them not edge-to-edge, which are missing from their lists, shown in Figures 2(a), 4(l) and 4(n). These additional tilings were used as designs for certificates presented to those receiving awards in the 2001 Problems Drive [16]. This article presents the full enumeration, with an outline of how it may be verified.

First we consider how the uniform tilings without star polygons may be enumerated. If $k$ regular polygons with $n_1, \ldots, n_k$ edges respectively meet at a vertex, we must have

$$\sum_{i=1}^{k} \frac{n_i - 2}{n_i} = 2.$$

Clearly $3 \leq k \leq 6$ and for each $k$ it is easy to determine the finitely many solutions. There are 17 possible choices of the $n_i$, where different orders are not counted as distinct; where different cyclic orders are counted as distinct (but the reversal of an order is counted as the same as that order), this yields 21 possible species of vertices. Some of these cannot occur in any tiling by regular polygons at all; for example, $3\,.\,7\,.\,42$ is the only possibility involving a heptagon, and this would mean that triangles and 42-gons must alternate around the heptagon, which is impossible since 7 is odd. This leaves 15 species that can occur in tilings by regular polygons. Of these, 11 yield the uniform tilings shown in Figure 1; it turns out that each yields a unique uniform tiling.[2] The vertex $4\,.\,8^2$ can only appear in the uniform tiling it generates, and not in any other tiling by regular polygons, since it is the only one of the 15 species containing an octagon; the others can appear in $k$-uniform tilings for suitable $k$ (and, for sufficiently large $k$, all species can appear together in one tiling).

Suppose now we consider tilings involving regular star polygons. In addition to ordinary regular polygons, a vertex of such a tiling may have points and dents of star polygons. We only consider uniform tilings, so all vertices are alike. Observe that no vertex can have two dents present, and two star points cannot be adjacent at a vertex. Also, since the tiling is supposed to contain some star polygon, it is easy to see that some vertex, and so all vertices, must have a star point. (For, if any star point is not a vertex, it fills a dent of a second star;

[2]Properly, it is necessary to show that each of the tilings in Figure 1 actually exists, since it is easy to draw what look like tilings by regular polygons but are actually fakes with polygons that are not exactly regular; examples of such drawings may be found in children's colouring books. It is not quite trivial that 'local' existence of the tilings implies global existence, but we do not discuss existence proofs further here.

but then the points of that star on either side of the filled dent must lie at vertices of the tiling.)

First we dispose of the tilings that are not edge-to-edge. This means that a vertex lies part way along the edge of some polygon. The vertex cannot have a dent, or two adjacent star points, but it must have a star point. Thus it has present either two ordinary regular polygons (one a triangle, the other a triangle, square or pentagon), with at least one star point (on one side, or between the polygons), or one regular polygon with star points on either or both sides. Bearing in mind that there cannot be a vertex at a dent, a careful analysis of cases (which the reader is encouraged to verify; it is convenient to start by showing that the vertex must lie on the edge of an ordinary regular polygon, not a star) shows that the possible tilings are those of Figure 2. The new one (used in the Problems Drive certificate for silliest answer) is Figure 2(a).

Having found those tilings, we need now only consider edge-to-edge tilings, in which all polygons will have the same edge length. In general the analysis of these is more systematic than that of the tilings that are not edge-to-edge. Because it essentially consists of analysis of many cases, most of the details are not presented here but are left to the reader, who will need to draw a large number of little diagrams for the various cases.

It is convenient to separate the cases where some dent lies at a vertex of the tiling (so, since the tiling is uniform, all vertices have a dent) from those where no dent is a vertex. If some dent is a vertex, it cannot be filled entirely by points of stars, so the polygons in the dent are $k$ (for some integer $k$) or $3\,.\,3$, $3\,.\,4$ or $3\,.\,5$, and each case is considered in turn, yielding the tilings in Figure 3 (four of which actually show an example of an infinite family of tilings).

Now suppose that no dent is a vertex. Considering the possible vertex figures, clearly no two star points can be adjacent; since no dent is a vertex, no point can lie between two regular polygons with different numbers of edges; and if a point of a star lies between two regular polygons with the same number of edges, their next vertices must fill its dents exactly. This means that there must be two adjacent regular polygons with the same number of vertices, separated by the point of a star, since we are only looking for tilings which do involve star polygons. This leads to the table (Table 1) of cases for the sequence of regular polygons (ignoring the star polygons). The new tilings are Figure 4(n) (used for the certificate for the winners) and Figure 4(l) (used for the certificate for the wooden spoon).

By way of example, consider the case $k^2$ ($k \geq 3$). If there is a single star point at the vertex, say $s_\alpha$, we have $\alpha = 4\pi/k$ and $2(s-1)\pi/s - \alpha = \pi + 2\pi/k$, so $2/s + 6/k = 1$. There is a combinatorial constraint that 3 divides $k$, and the integer solutions yield Figures 4(e), (f) and (g). If there are two star points at the vertex, say $s_\alpha$ and $t_\beta$, both dents are filled by the $k$-gon vertex, so $2(s-1)\pi/s - \alpha = \pi + 2\pi/k = 2(t-1)\pi/t - \beta$; thus $\alpha = (1 - 2/s - 2/k)\pi$ and $\beta = (1 - 2/t - 2/k)\pi$, yielding $1 = 4/k + 1/s + 1/t$. Combinatorially, $k$ is even, and if $s \neq t$ then 4 divides $k$; the solutions subject to these constraints yield Figures 4(h) to (k).

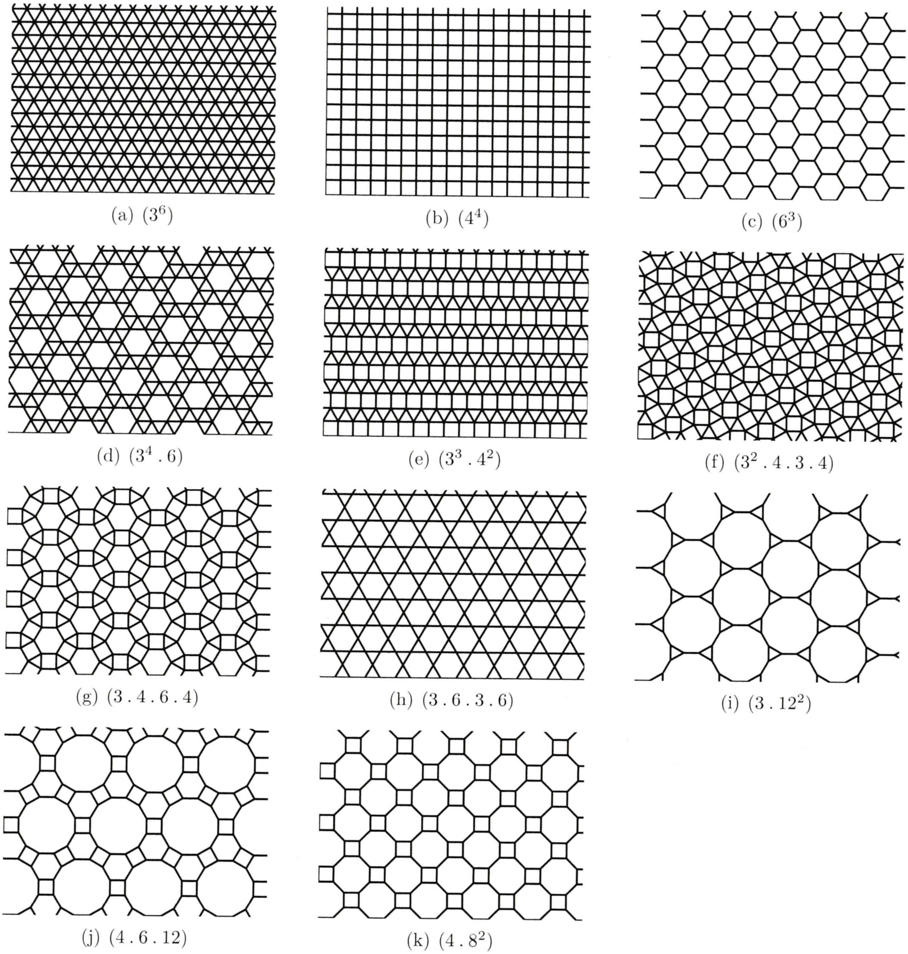
A natural extension of this work would be to enumerate 2-uniform edge-to-edge tilings by regular polygons and regular star polygons. I have done some work towards this, but completing such an enumeration by hand would be substantially time-consuming and error-prone. As Kepler's diagrams include various examples of $k$-uniform tilings (and of small patches that can plausibly be extended to such tilings) such enumeration could be seen as continuing the systematic completion of Kepler's work. It would be interesting to develop a sufficiently systematic method of finding $k$-uniform tilings involving star polygons that such tilings could be searched for by computer. Even for tilings not involving star polygons, more efficient enumeration algorithms might be able to extend the enumeration far beyond that of [15]. A deeper problem would be to attempt to gain some understanding of the asymptotic behaviour of the number of $k$-uniform tilings of any variety.

Another direction would be to attempt to determine which polygons can occur at all in edge-to-edge tilings by regular polygons and regular star polygons, with no uniformity conditions. For example, can regular $n$-gons or $n$-stars with $n > 18$ occur? Can regular heptagons occur?

When considering tilings that are not edge-to-edge, Grünbaum and Shephard present as a research exercise determining the 2-uniform tilings, not involving star polygons, with the additional constraint that the tilings be equitransitive (i.e., that the symmetries act transitively on each congruence class of tiles). This problem could be attempted with or without that constraint, or with star polygons allowed. An attempt could also be made at a search algorithm for such tilings that could be implemented on computer.

| Vertex sequence | Tilings (Figure 4) |
|---|---|
| $3^5$ | (a) |
| $3^4 . 4$ | (b) |
| $3^4 . 5$ | None |
| $3^4$ | None |
| $3^3 . k \quad (k \geq 4)$ | None |
| $3^3$ | None |
| $3^2 . k \quad (k \geq 4)$ | None |
| $3^2 . 5^2$ | None |
| $3^2 . a . b \quad (4 \leq a \leq 5 \leq b, a < b)$ | (c), (d) |
| $3^2 . 4^2$ | None |
| $k^2 \quad (k \geq 3)$ | (e) to (k) |
| $3 . k^2 \quad (k \geq 4)$ | (l), (m) |
| $3 . 4^3$ | (n) |
| $3 . 4^2 . 5$ | None |
| $4^3$ | (o) |
| $4^2 . k \quad (k \geq 5)$ | None |
| $4 . k^2 \quad (5 \leq k \leq 7)$ | (p) |
| $5^3$ | (q) |
| $5^2 . k \quad (6 \leq k \leq 9)$ | None |
| $5 . 6^2$ | None |

Table 1: Cases where no dent is a vertex



(a) $(3^6)$ (b) $(4^4)$ (c) $(6^3)$

(d) $(3^4 . 6)$ (e) $(3^3 . 4^2)$ (f) $(3^2 . 4 . 3 . 4)$

(g) $(3 . 4 . 6 . 4)$ (h) $(3 . 6 . 3 . 6)$ (i) $(3 . 12^2)$

(j) $(4 . 6 . 12)$ (k) $(4 . 8^2)$

Figure 1: Archimedean tilings

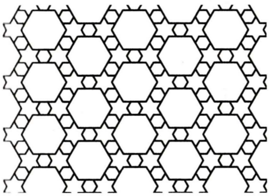
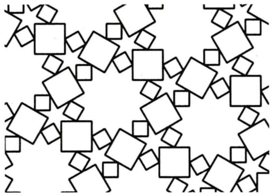
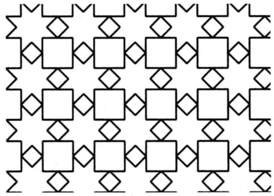
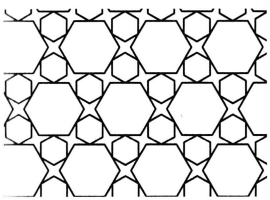
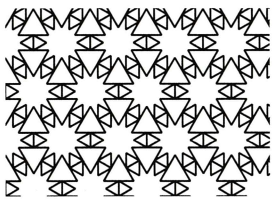
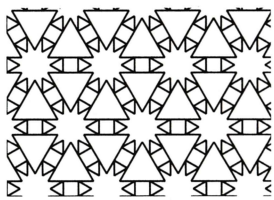

(a)

(b)

(c)

(d)

(e)

(f)

Figure 2: Uniform tilings involving star polygons which are not edge-to-edge

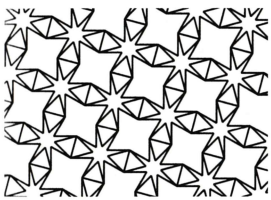
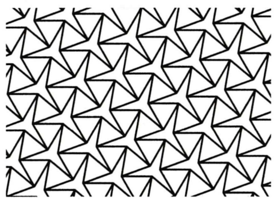
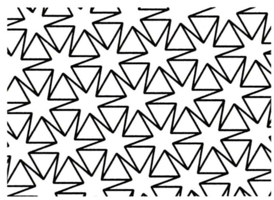
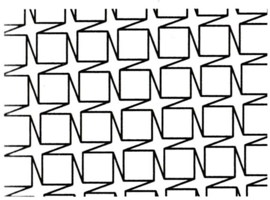
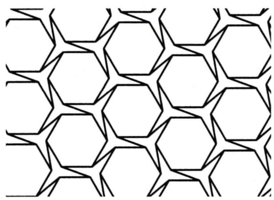
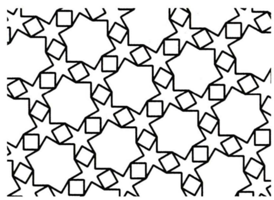
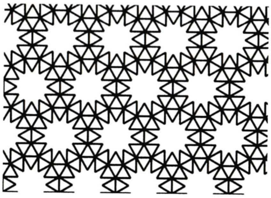
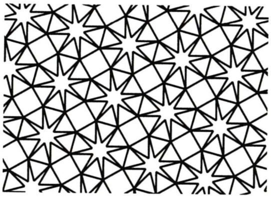

(a) $(3^2 . 8^*_{\pi/12} . 4^{**}_{\pi/3} . 8^*_{\pi/12})$

(b) family $(3 . 3^*_\alpha . 3 . 3^{**}_\alpha)$

(c) family $(3 . 6^*_\alpha . 6^{**}_\alpha)$

(d) family $(4 . 4^*_\alpha . 4^{**}_\alpha)$

(e) family $(6 . 3^*_\alpha . 3^{**}_\alpha)$

(f) $(4 . 6^*_{\pi/6} . 6^{**}_{\pi/2} . 6^*_{\pi/6})$

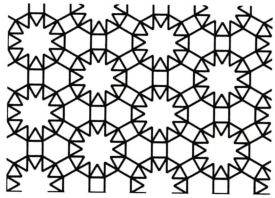
Figure 3: Uniform tilings in which some dent is a vertex



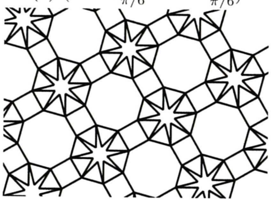
(a) $(3^3 . 12^*_{\pi/6} . 3^2 . 12^*_{\pi/6})$



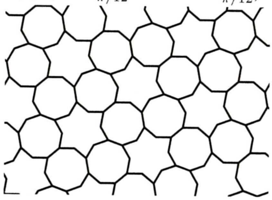
(b) $(3^2 . 8^*_{\pi/12} . 3 . 4 . 3 . 8^*_{\pi/12})$



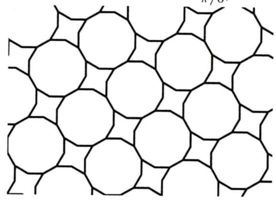
(c) $(3 . 4 . 6 . 3 . 12^*_{\pi/6})$



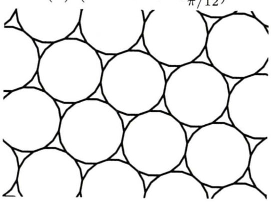
(d) $(3 . 4 . 8 . 3 . 8^*_{\pi/12})$



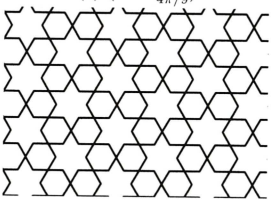
(e) $(9^2 . 6^*_{4\pi/9})$



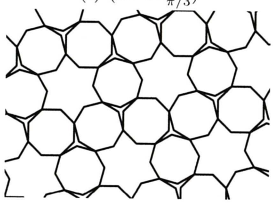
(f) $(12^2 . 4^*_{\pi/3})$



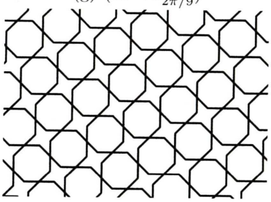
(g) $(18^2 . 3^*_{2\pi/9})$



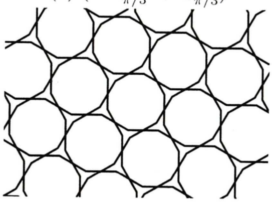
(h) $(6 . 6^*_{\pi/3} . 6 . 6^*_{\pi/3})$



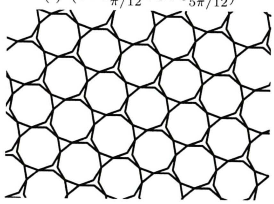
(i) $(8 . 3^*_{\pi/12} . 8 . 6^*_{5\pi/12})$



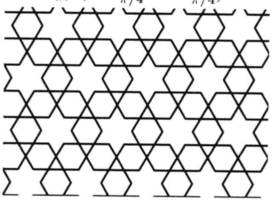
(j) $(8 . 4^*_{\pi/4} . 8 . 4^*_{\pi/4})$



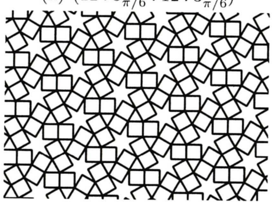
(k) $(12 . 3^*_{\pi/6} . 12 . 3^*_{\pi/6})$



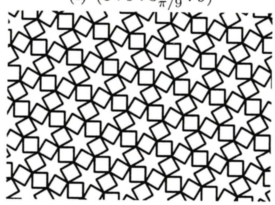
(l) $(3 . 9 . 3^*_{\pi/9} . 9)$



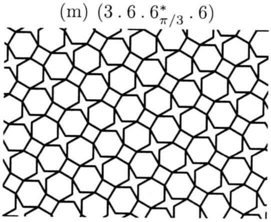
(m) $(3 . 6 . 6^*_{\pi/3} . 6)$



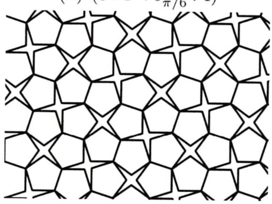
(n) $(3 . 4^2 . 6^*_{\pi/6} . 4)$



(o) $(4 . 6^*_{\pi/6} . 4 . 6^*_{\pi/6} . 4 . 6^*_{\pi/6})$



(p) $(4 . 6 . 4^*_{\pi/6} . 6)$



(q) $(5^2 . 4^*_{\pi/10} . 5 . 4^*_{\pi/10})$

Figure 4: Uniform tilings in which no dent is a vertex

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*[Ed: In December 2003, the Archimedeans held a Puzzle Hunt, which, as is traditional, involved dashing round several rooms to solve various problems. Again as is traditional, a prize was awarded for the best 'solution' to the story-writing puzzle, which this year asked for "a story of at least 223 words that has a theme related to both mathematics and Christmas"; this prize went to Stephen Burgess, and his story is printed below.]*

# How Mathematics Saved Christmas

## Stephen Burgess

As the stars twinkled in the moonlit night, Santa Claus loosened the grip on the reindeer. "Why should I give toys to children who are ignorant of mathematics?", he exclaimed, in a raucous voice, quite unbecoming of the traditional view of Santa. And so he devised a cunning test to determine the child's proficiency—he would wake each up in turn and ask them a simple sum. However, Fluffy the Elf did not think this was a fair test, so, in the disguise of a young child, he lay in wait. As Santa processed the children, turning all of those who answered incorrectly into garden gnomes, he eventually came upon Fluffy. Fluffy shook in terror at the prospect of the question. Santa, evilly grinning at the thought of all of the new gnomes he had acquired, boomed "What is the shortest distance between two points on the surface of a sphere?". Fluffy's eyes brightened. "In a spherical or Euclidean metric?", he replied. Santa shrugged his shoulders. "I don't know." The gnome machine whirred into life and Santa was transformed into a gnome. He cried out and a red, chubby, bearded gnome came out the other end. Fluffy and the elves celebrated with mulled wine and took over Lapland, giving presents to everyone. And that is why, children, all gnomes are now red and have large beards, and Santa doesn't really exist any more.

# Request for Reminiscences

Jordan Skittrall would very much like to hear from anybody with memories of the Archimedeans (or the College Societies) in years gone by. It is intended that an edited selection of such correspondence should be used to form the basis of an article in a future issue of Eureka, or perhaps even a small booklet.

The Society Archivist, Dr Joseph Myers, would be interested in hearing from anybody who has found archives from the Archimedeans or College Societies in their possession (in particular some minutes books are not in the central archive), or from anybody who could assist with filling in the details of the Archimedeans' talks held in the Lent Term 1983 or the Trinity Mathematical Society's talks held in 1994–5.

Correspondence may be sent to them at the Society address (inside the front cover).

# Book Reviews

## Flexagons Inside Out

by **Les Pook**, Reviewed by **Stephen Burgess**

Cambridge University Press, 2003,
ISBN 0-521-81970-9

For those of you who do not know, a flexagon is a hinged polygon with the property that, on "flexing", a different face may be revealed; for instance in a trihexaflexagon, at any time only the top and bottom faces are visible, but on flexing, a third face becomes visible. These cute oddities were first discovered in the 1960s by Martin Gardner, so it is some surprise that a book on them has taken this long to arrive. Although, in the author's words, "there is an infinite number of possible types of flexagons so no book on flexagons can be comprehensive", Pook has here included much of the currently known information on flexagons, together with some new material on flexahedra, their 4D analogues. Although the level of mathematical detail in the book will be insufficient for some, this is primarily a "make-and-do" topic, which is reflected in the large number of diagrams in the book, making it an excellent resource for anyone with little previous knowledge to understand the basics, but with enough detail to satisfy the interest of all but the most ardent mathmos.

## The Changing Shape of Geometry: Celebrating a Century of Geometry and Geometry Teaching

Edited by **Chris Pritchard**, Reviewed by **Jenny Gardner**

Cambridge University Press, 2003,
ISBN 0-521-53162-4

This is a collection of short articles by a variety of authors, which makes it great to dip into when you have a minute to spare. The book is based around thirty "desert island theorems": results that eminent mathematicians such as Coxeter, Hofstadter, Atiyah and Singh have felt would keep them occupied if they were stranded on a desert island. These results, some well-known and others less familiar, are in general attractive and surprising, and are presented elegantly and clearly. I can see that they would provide adequate food for thought. There are also articles, many taken from the Mathematical Gazette, on several areas of geometry, including "The History of Geometry" and "The Golden Ratio". The latter section is particularly interesting, as it provides many surprising incidences of the golden ratio in a variety of geometrical diagrams. The sixth and last section of the book is "The Teaching of Geometry", which is probably less appealing to students. Nevertheless, I would recommend this book to anyone who likes geometry; whatever level of experience you have with geometry, you are likely to enjoy the presentation of such a variety of geometrical ideas by the many prestigious authors.

# From the Mathematical Innovations Catalogue

Chris Cummins

- Klein salad-dressing bottle. Keep oil on the inside, vinegar on the outside.
- Escheristic perpetual-motion machine. The ball rolling down an infinite slope generates enough energy to power a light bulb. New! Uphill version. Uses 2 1.5V AA batteries per day. The ideal gift for someone you dislike.
- Quantum surfboard. Warning: do not use on unrestricted wavefunctions.
- Random walk generator. Comprises stereotypical mathmo, half-pint of lager.
- COMING SOON: 3D random walk generator. Also includes centrifuge, trampoline. Accessories:
  - Reflecting barriers
  - Absorbing barriers
  - Extra-absorbing barriers, for mopping up resulting spills
- Dr Leader's Evil Adversary self-defence kit. Sprays arbitrarily small epsilons over a range of delta metres.
- Anthropomorphiser. Ascribes human qualities and emotions to functions, sets, numbers etc. Not for use on mathmos.
- Epsilon magnifier. Sick of struggling with tiny epsilons? The revolutionary new epsilon magnifier simplifies analysis by increasing all epsilons to values $> 1$.
- Book of mispelt adz for pedents. Hours of fun for mathmos.
- Matching set of pathological cases. For the more experienced traveller, save money with our nowhere-dense set of luggage.
- Calculus-removing toothpaste. Guaranteed opaque.

# Solutions to the Problems Drive

Toby Kenney and Paul Russell

1. 16

2.

| | | | | | |
|---|---|---|---|---|---|
| $^{1}1$ | 5 | $^{2}8$ | | | $^{3}5$ |
| 1 | | $^{4}8$ | 1 | $^{5}2$ | 5 |
| $^{6}6$ | $^{7}3$ | 7 | | $^{8}9$ | 7 |
| | 3 | | $^{9}7$ | 2 | 9 |
| $^{10}2$ | 5 | $^{11}6$ | | 8 | |
| 9 | | $^{12}7$ | 9 | 2 | 1 |

3. (I) x (Alan Baker, 19$^{\text{th}}$ August 1939)
   (II) iv (Pierre de Fermat, 17$^{\text{th}}$ August 1601)
   (III) vii (Stefan Banach, 30$^{\text{th}}$ March 1892)
   (IV) ii (Diophantus, c.200 A.D.)
   (V) ix (Alexander Grothendieck, 28$^{\text{th}}$ March 1928)
   (VI) viii (William Hodge, 17$^{\text{th}}$ June 1903)
   (VII) i (Eratosthenes, 276 B.C.)
   (VIII) xii (Timothy Gowers, 20$^{\text{th}}$ November 1963)
   (IX) iii (Nicolaus Copernicus, 19$^{\text{th}}$ February 1473)
   (X) vi (Augustin-Louis Cauchy, 21$^{\text{st}}$ August 1789)
   (XI) v (Isaac Newton, 25$^{\text{th}}$ December 1642)
   (XII) xi (Andrew Wiles, 11$^{\text{th}}$ April 1953)

4. 337 (there are other acceptable solutions)

5. $\frac{7}{12}$

6. 5$^{\text{th}}$

7. (i) 2025 (squares of triangular numbers)
   (ii) 387420489 ($n^n$)
   (iii) 58786 (Catalan numbers)
   (iv) 181 (squares + primes)
   (v) 2336 (alternately add $n^3$ and subtract $n^2$)
   (vi) 1 (number of distinct groups of order $n$ up to isomorphism)

8. (i) $3^e$
   (ii) $e^3$
   (iii) $3^{\frac{11}{4}}$
   (iv) $(\frac{11}{4})^3$
   (v) $3^{2\sqrt{2}}$

(vi) $\pi^e$
(vii) $(2\sqrt{2}\,)^3$
(viii) $(\sqrt{10}\,)^e$
(ix) $e^\pi$
(x) $\pi^{\frac{11}{4}}$
(xi) $e^{\sqrt{10}}$
(xii) $(\sqrt{10}\,)^{\frac{11}{4}}$
(xiii) $(\frac{11}{4})^\pi$
(xiv) $(\frac{11}{4})^{\sqrt{10}}$
(xv) $\pi^{2\sqrt{2}}$
(xvi) $(\sqrt{10}\,)^{2\sqrt{2}}$
(xvii) $(2\sqrt{2}\,)^\pi$
(xviii) $(2\sqrt{2}\,)^{\sqrt{10}}$

9. 6, 15, 51, 57, 60, 66, 75, 105, 150, 156, 165, 255, 501, 507, 510, 516, 525, 552, 558, 561, 567, 570, 576, 585, 600, 606, 615, 651, 657, 660, 666, 675, 705, 750, 756, 765, 855

10. M=4, A=3, T=6, H=8, S=1, F=2, U=0, N=9

11. $\frac{\sqrt{3}}{\sqrt{2}}$, $\frac{1}{\sqrt{6}}$

12. 4, 8

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