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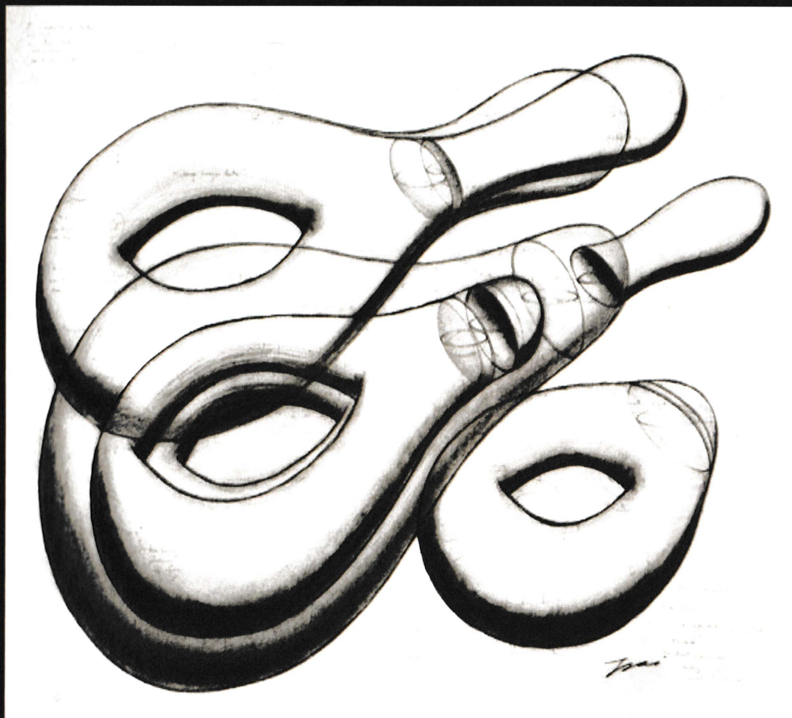

Ricci Flow with Surgery by Lun-Yi Tsai

Every simply connected closed 3-manifold is homeomorphic to S^3

Number 59 March 2008

front cover

The cover is Lun-Yi Tsai's depiction of the process of "surgery" of the Ricci flow used by Grigoire Perelman in his proof of the Poincare conjecture.

The piece is titled "Ricci flow with surgery", 2007, drawn with charcoal and graphite on paper, 26.5 by 40 inches.

More information is available at http://www.lunyitsai.com/

eureka

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The Pumping Lemma

Martin Orr, Trinity College

1 Introduction

The theory of formal languages has its roots in logic and computer science,where it was applied to very abstract languages, but much of the early development was also carried out in linguistics, in the hope of applying it tonatural languages such as English. English, programming languages such as C and algebraic equations are all examples of the languages to which we might try to apply these techniques.

However, we will adopt a much more general definition: start with any finite set of symbols, an ***alphabet***. Then a ***string*** is any finite sequence of these symbols (including the sequence of length zero, represented by ϵ) and a ***language*** is any set of strings.

Writing algorithms to parse languages is an important practical problem; it is very difficult when applied to English, while of course programming languages are designed to make this easy. It also leads to deep theoretical problems about which languages can be recognised algorithmically: from the above definition it is easy to show that there are uncountably many languages. But there are only countably many algorithms, so there exist languages for which there is no algorithm which decides whether or not a string is in the language.

2 Context-Free Languages

Many classes of languages have been defined, based on the complexity of the algorithms required to recognise them. In this article, we will consider ***context-free languages*** or *CFLs*. These can be naturally described by ***context-free grammars*** (*CFGs*), and there exist efficient general algorithms for parsing large classes of CFLs. Consequently most programming languages are CFLs. They do not include all simple languages however: the main aim of this article will be to prove that the language $L_1 = \{a^n b^n c^n \mid n \geq 1\}$ (i.e. all strings with some number of *a*s, followed by an equal number of *b*s, followed by an equal number of *c*s) is not context-free.

A simple example of a context-free grammar might look like

$$
\begin{gathered}
S \rightarrow E=E \\
E \rightarrow E+E \\
E \rightarrow E-E \\
E \rightarrow x \\
E \rightarrow y \\
E \rightarrow z
\end{gathered}
$$

This grammar defines a language L_2 of simple equations in the variables x, y, and z. The uppercase letters are not part of the alphabet that can appear in the language: they are just used as placeholders within the grammar. The symbol S represents an equation and E an expression. Each rule (called a ***production*** means that the symbol on the left can be replaced by the string on the right. We start from the ***start symbol*** S and apply the rules in some order until we are left with only lowercase letters and punctuation symbols.

So for example, the string $x - y = x + z$ is an equation according to this grammar because it can be obtained by the following steps:

$$
\begin{gathered}
S \Rightarrow E=E \Rightarrow E-E=E \Rightarrow E-E=E+E \\
\Rightarrow x-E=E+E \Rightarrow x-y=E+E \\
\Rightarrow x-y=x+E \Rightarrow x-y=x+z
\end{gathered}
$$

A sequence of replacements like this is called a ***derivation***.
The symbols in the language itself (lowercase letters) are called ***terminals*** and those used on the left-hand side of productions (uppercase letters) are ***non-terminals***. Each production must have just a single non-terminal on the left. This is why these are called context-free grammars: the possible replacements for a non-terminal do not depend on the context in which it appears.

Another example of a context-free language is the set $L_3 = \{a^n b^n \mid n \geq 1\}$.
It is generated by the CFG

$$
\begin{aligned}
&S \rightarrow aSb \\
&S \rightarrow ab
\end{aligned}
$$

3 Regular Languages

Before considering how to prove that languages are not context-free, we consider a simpler case: regular languages. A ***regular language*** is a context-free language in which all productions have one of the forms

$$A \to bC$$
$$D \to e$$
$$F \to \epsilon$$

In other words a regular language can simply be analysed one terminal at a time from left to right, keeping track of which non-terminals could match the rest of the string. Note that there are only finitely many possible sets of non-terminals, so we only need to "remember" which of a finite number of states we are in (a computing device which works like this is called a ***finite state automaton***

A simple example of a regular language is $L_4 = \{a^n b^m \mid n, m \geq 1\}$. This is generated by the regular grammar

$$S \to aS$$
$$S \to aB$$
$$B \to bB$$
$$B \to b$$

4 A non-regular language

Consider the language $L_3 = \{a^n b^n \mid n \geq 1\}$ which we have shown to be context-free. The grammar given in section 2 is not regular, but this does not rule out the possibility that there is some other regular grammar which generates this language.

However, we observed above that it is possible to determine whether or not a string belongs to a particular regular language by working from left to right, and using only a ``memory" with a finite number of states. On the other hand, to check whether or not a string is in L_3, we have to count the number of as, and then check that the number of bs is the same; but this requires us to remember the value of n, which could have infinitely many different values (any positive integer).

We now attempt to make this intuition rigorous. Note that if we set an upper bound for the value of n, then the argument fails; and indeed the language that results is regular. So we should look at what happens when n is large. When n is larger than the number of sets of non-terminals, then there must be two different points at which the sets which could match the rest of the string are the same. These two points divide the string into three parts, say xyz. Since z matches the same non-terminals as yz, we would not have noticed if y was missing from the string. So $xz \in L$ as well. Furthermore, we would not have noticed if y was repeated, and this leads us to the

Pumping Lemma for Regular Languages
For any regular language L, there exists a natural number p such that any string in L with length at least p can be written in the form xyz where y has length greater than zero and $xy^m z$ is in L for all $m \geq 0$.

The proof was in the previous paragraph (let p be 2 to the power of the number of non-terminals in the language).

We should check that this lemma does indeed prevent L_3 being a regular language. Consider the string $a^p b^p$ which certainly has length at least p. When this divided into three pieces, xyz, either y contains only as or only bs, in which case xz no longer has the same number of as and bs, or else y contains both as and bs, in which case $xyyz$ is $a^p b^i a^j b^p$ for non-zero i and j, so is not in L_3 .

5 Pumping lemma for context-free languages

A context-free language does not have the simple left-to-right structure of a regular language, but a natural way of analysing them is to consider a ***parse tree***.

For example, a parse tree for the string $x - y = x + z$ in L_3 appears in Fig 1. Each non-terminal appears above the symbols which replace it in the derivation, and the terminals appear along the bottom.

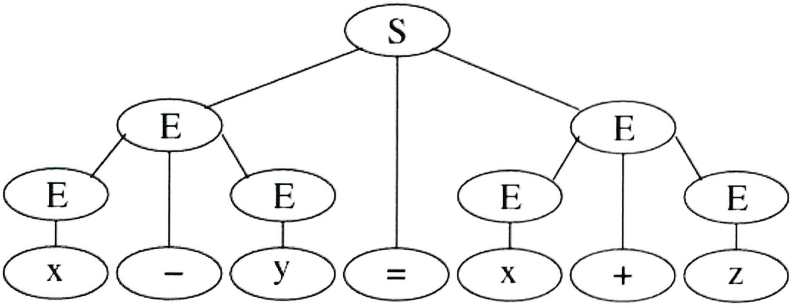

Figure 1: Parse tree for $x - y = x + z$

Using parse trees, we create a necessary condition for CFLs, similar to the pumping lemma for regular languages, showing that there are pieces of the string which can be "pumped" or repeated arbitrarily many times, while remaining in the language.

Pumping lemma for Context-free Languages.
For any context-free language L, there exists a natural number p such that any string in L with length at least p can be written in the form uvxyz where vy has length greater than zero and $uv^m xy^m z$ is in L for all $m \geq 0$.

Proof. Let the number of non-terminals in L be n and the length of the RHS of the longest production be l. Let $p = l^n + 1$.

Let w be a string in L with length at least p. Consider a parse tree for w. There may be more than one possible parse tree; choose one with the minimal number of nodes. If all branches in this tree have depth at most n, then there can be at most l^n non-terminals in w. So in fact some non-terminal q in w is at the end of a branch of depth greater than n.

This means that the branch leading to q must contain some non-terminal N twice. Now we can write w in the form $uvxyz$, where the higher of these Ns matches vxy and the lower matches x. If the strings v and y both have length zero, then we could just remove one N and all the nodes between the Ns and get another parse tree for w, contradicting the assumption that our tree had the minimal number of nodes. So at least one of v and y has length greater than zero.

This tells us that we can derive $S \Rightarrow uNz$, $N \Rightarrow vNy$ and $N \Rightarrow x$ in L. But by applying the second of these recursively, we get $N \Rightarrow uv^m xy^m z$ for all natural numbers m.

So $S \Rightarrow uv^m xy^m z$ for all $m \geq 0$.

□

From here it is fairly straightforward to show that L_1 is not context-free: Suppose we had a CFG for L_1 . Then find the pumping length p as in the lemma, and consider $w = a^p b^p c^p$.

Since this is longer than p, we can split it into $uvxyz$ as in the lemma. Since $uxz \in L_1$, it contains equal numbers of as, bs and cs. So vy also contains equal numbers of as, bs and cs (and this number must be more than zero), so at least one of v and y contains more than one distinct letter. So $uvvxyyz$ must contain a b before an a or a c before a b, and is not in L_1 . Hence L_1 does not satisfy this pumping lemma, so is not context-free.

6 Conclusion

Context-free grammars are useful because of the level of generality they provide in describing sets of strings: restricted enough that there are reasonably efficient general purpose algorithms for determining whether a string is generated by a given CFG, yet general enough to describe a lot of the useful languages. In addition, they not only tell us whether a string is in the language or not, but also can describe the structure of the string: the parse tree in the equation example above matches the way the equation is made up of smaller pieces. Indeed, CFGs are an essential tool in describing natural languages in linguistics, and it is an unanswered question as to whether or not they are sufficient for this purpose.

However, the { $a^n b^n c^n$ } example shows that there are some apparently very simple languages which are not context-free. Regular and context-free grammars are two levels in the ***Chomsky hierarchy***, a hierarchy of grammars described by the linguist Noam Chomsky in the 1950s. More general classes are context-sensitive grammars and unrestricted grammars; the latter can describe all languages for which there is any algorithm to accept them.

There are many questions we can ask about these classes of languages. Some of these ask about a particular example of a language where it fits in the hierarchy. The pumping lemma presented here is one tool useful in answering this; note that it is a necessary but not a sufficient condition for a language to be context-free. If you fancy a challenge, you might like to try to show that $\{a^n b^n c^m \mid n, m \geq 1 \text{ and } n \neq m\}$ satisfies the conditions of the pumping lemma but is not context-free.

As well as these relatively mundane problems, grammars lead to deep questions in the theory of computation. For example, given two context-free grammars, do they generate the same language? There is no general algorithm which can answer this. We can consider a range of similar questions about different classes in the Chomsky hierarchy and other related grammars. Sometimes there are known algorithms; many such questions are provably undecidable; others remain unsolved.

Shuffle: a game of card arrangement

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Introduction

In this article, we introduce a card game based on the concept of permutation. In the card game, a player is given a predetermined number of cards in incremental order, and following a predefined condition which requires the cards to be rearranged, the objective is to find an ordered arrangement of cards which when executed according to the predefined condition with the cards facing down, will make the cards appear in incremental order. We illustrate this by the following example.

Example of a Game

Predetermined number of cards: Let's say at the beginning of the game, 13 cards (*A*, *2*, *3*, …, *K*) are given to the player.

Predefined condition for arrangement: Let's say the condition is to use the English spelling of the cards.

Objective: The player therefore has to rearrange the cards according to the condition. The arrangement has to be done in such a way that when the cards are facing down and cards are called out according to the condition, the cards will be revealed in incremental order.

Solution: Once arranged, and with the cards facing down, the player will say "A" followed by "C" and then "E" (which gives the word "ACE"), each time shifting the uppermost card to the bottom of the deck, and the card *A* will be revealed

The process continues through spelling the individual cards in English and in the end the cards are revealed in incremental order from *A* to *K*.

Finding the Solution

The arrangement that enables the cards to be revealed in incremental order can be found in a simple way by drawing an elimination and placement table (Figure 2).

ini	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**J**	**Q**	**K**
	0	0	0	1	0	0	0	2	0	0	0	0	0
	3	0	0	X	0	0	4	X	0	0	0	0	5
	X	0	0	X	0	6	X	X	0	0	0	0	X
	X	0	7	X	0	X	X	X	0	0	0	0	X
	X	8	X	X	0	X	X	X	0	0	0	9	X
	X	X	X	X	0	X	X	X	0	0	10	X	X
	X	X	X	X	0	X	X	X	0	0	X	X	X
	X	X	X	X	0	X	X	X	J	0	X	X	X
	X	X	X	X	0	X	X	X	X	0	X	X	X
	X	X	X	X	0	X	X	X	X	0	X	X	X
	X	X	X	X	Q	X	X	X	X	K	X	X	X
sol	**3**	**8**	**7**	**1**	**Q**	**6**	**4**	**2**	**J**	**K**	**10**	**9**	**5**

Key
0 – denotes the uppermost card that is moved to the bottom of the deck of cards
X – denotes a card that has been revealed and therefore not counted further

Figure 2. The elimination and placement table.

Conclusion

The game can be played with a variety of predetermined number of cards and predefined condition. For example, if the number of cards is 13 (*A* to *K*), and the condition is interval of one (i.e. moving one card to the bottom of the deck and revealing the next card), the solution is: **7 A Q 2 8 3 J 4 9 5 K 6 10**.

Thus, players can state the number of cards used and the condition applicable, and throw a challenge at each other to determine the correct arrangement.

We list here, some examples of conditions (where n is any positive integer) that can be used:

(i) fixed intervals of size n

(ii) alternate intervals, e.g. n and n+1 alternately (the alternate intervals can be in triples, quadruplets etc.)

(iii) increasing intervals, e.g. n, n+1, n+2,...

(iv) variable intervals, e.g. n, n+2, n+4, n+3, n+1

In fact, players can simply think up any form of intervals (or even use randomly generated positive integers!).

The challenge would inevitably be who is *the fastest to get the correct arrangement* based on the number of cards and conditions used.

And of course, part of the challenge is *not* to use the elimination and placement table in finding the correct arrangement. Can you find a way how to do this?

Enjoy the card-based Rubik's cube!

Feynman's Proof of the Law of Ellipses

Viktor Blåsjö

This article presents a brief exposition of Richard Feynman's proof that planets move in ellipses around the sun. Feynman presented this proof in his Caltech lectures in 1963. It is published in [1], which contains a verbatim transcription of the lecture and extensive commentary, making it almost two hundred pages long. The proof is in essence short and sweet, however, so I felt that a succinct exposition might be useful. As a preliminary we must first prove that planets sweep out equal areas in equal times. For this we follow Newton's proof, [2], book I, sec. II, prop. I, thm. I.

Theorem: *Planets sweep out equal areas in equal times.*

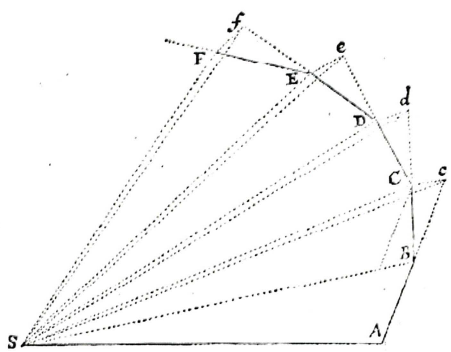


Figure 1

Proof. In an infinitely small period of time the planet has moved from A to B. If we let an equal amount of time pass again then the planet would continue to c if it was not for the gravity of the sun, which intervenes and deflects the planet to C. Since the time it takes for the planet to move from B to C is infinitely small, the gravitational pull has no time to change direction from its initial direction BS, thus causing cC to be parallel to BS.

Therefore the triangles BcS and BCS have equal areas, since they have the same base, BS, and equal height. Also, the triangles ABS and BcS have the same area since they have equal bases, AB and Bc, and the same height. So the triangles ABS and BCS have equal areas—the planet has swept out equal areas in two consecutive, infinitely small time intervals. So the planet never changes the rate at which it sweeps out area and the theorem follows. At this stage Feynman says: "Now the remaining demonstration is not one which comes from Newton, because I found I couldn't follow it myself very well, because it involves so many properties of conic sections. So I cooked up another one."

Theorem. *Planets move in ellipses.*

Proof. Cut the orbit into infinitesimal, equiangular pieces (as seen from the sun). Each little piece of the orbit corresponds to the velocity vector at that point. Draw a velocity diagram by moving all of these velocity vectors so that they have a common origin point.

Obviously, as we move around the orbit, the velocity vector will make one revolution around the origin. In fact, it will trace out a circle, as we shall now prove. The orbit is cut into infinitesimal triangles with equal angles at the sun, so clearly these triangles are similar with a scaling factor r, i.e. an area scaling factor r^2.

But time is the same as area, so time also varies as r^2.

The change in velocity in one of these pieces is $\text{force} \times \text{time} = \frac{1}{r^2} \times r^2$ which is independent of r, so the dv steps in the velocity diagram are all of equal size, and because of the equiangular division they all make equal angles with each other (dv parallel to PS), so the velocity vector does indeed trace out a circle, and the equiangular division of the orbit as seen from the sun translates to an equiangular division of this circle as seen from its center.

Of course, the center of the circle is not the origin of the velocity vectors; in particular, the velocity vector going through the center of the circle is the longest velocity

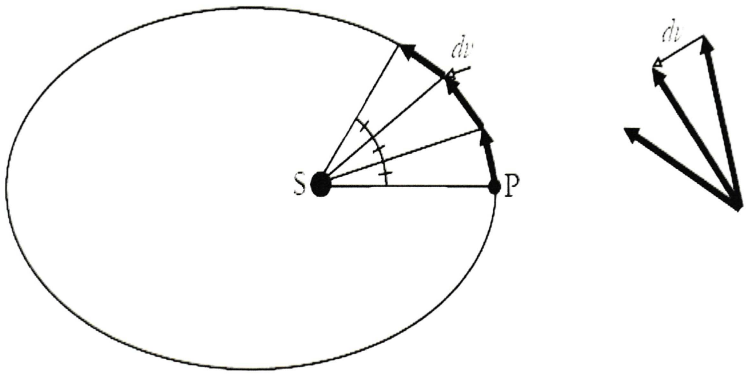


Figure 2

vector, so it corresponds to the position on the orbit closest to the sun (as is obvious by the law of equal areas). If we turn the orbit diagram so that this position is straight to the right of the sun, then the longest arrow in the velocity diagram points straight up, since the velocity vector drawn in the orbit diagram will of course be parallel to the tangent to the orbit.

When we have advanced a given angle beyond this starting point on the orbit (as seen from the sun), the corresponding velocity vector (i.e. essentially the tangent to the orbit at this point) is found by advancing the same angle in the velocity diagram (as seen from the center of the circle) and connecting this boundary point with the origin of the velocity vectors, and conversely. So the velocity diagram contains complete information about the tangents of the orbit, so it contains complete information about the orbit up to scaling.

So the problem becomes: for any velocity diagram, to recreate the orbit. To do this we turn the velocity diagram 90 degrees to the right. To recreate the orbit we must now find a curve that is always perpendicular to the velocity vectors. This can be done as follows. For any point p on the circumference of the velocity diagram circle, draw the line connecting it to the origin O of the velocity vectors and the line connecting it to the center C of the circle.

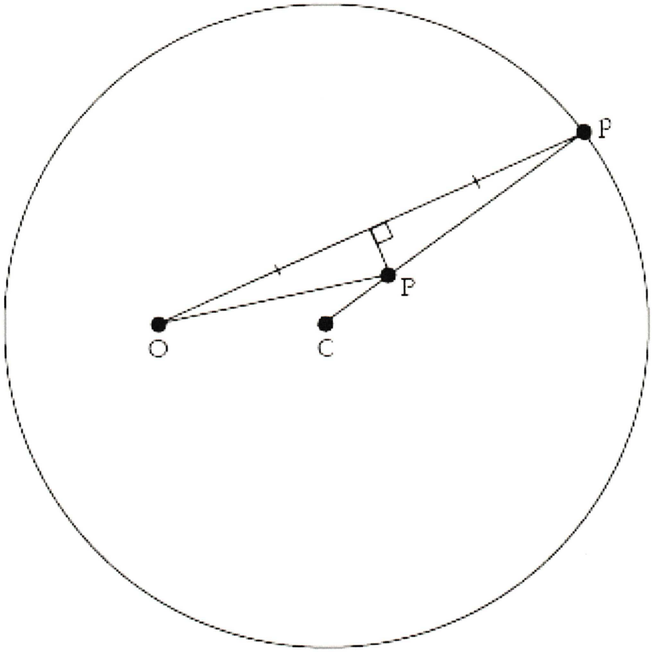


Figure 3

Mark the point P where the perpendicular bisector of Op cuts Cp as a point on the orbit. Now we prove that the orbit generated in this way, as p moves around the circle, is an ellipse.

The perpendicular bisector cuts the triangle OPp into congruent halves (SAS), making $OP = Pp$, so $CP + OP = CP + Pp$ = radius of the circle = independent of p, so P traces out an ellipse with foci C and O, and the perpendicular bisector is tangent to this ellipse (because all its other points are outside of the *ellipse because they have greater sum of distances to the foci),* as required.

□

References

[1] David L. Goodstein and Judith R. Goodstein (eds.), *Feynman's lost lecture: The motion of planets around the sun*, W. W. Norton & Company, New York, 1996.
[2] Isaac Newton, *Philosophiae naturalis principia mathematica*, 1687.

Adventures with Polynomials: A Criterion for Weil Numbers

David Loeffler, Trinity College alum.

ABSTRACT

We consider the following problem: given a polynomial P over the complex numbers, when is it the case that all of the roots of P have the same absolute value?

1 Introduction

Recently, while doing some calculations related to automorphic forms on a definite unitary group, I encountered the polynomial

$$P(x) = x^3 - \frac{-7+\sqrt{-259}}{8}x^2 + \frac{-7-\sqrt{-259}}{4}x - 8$$

A statement known as the generalised Ramanujan-Petersson conjecture implies that the roots of this polynomial should be ***Weil numbers***: a Weil number is an algebraic integer all of whose Galois conjugates have the same absolute value. In this case, I expected that all of the roots of this polynomial should have absolute value 2. So I wondered: how would one actually check this?

Of course, with today's computers it is a trivial matter to calculate the roots to any desired degree of accuracy; MAPLE opines that the roots are $1.606167789 + 1.191731945i$, $-0.7654506363 + 1.847724362i$, and $-1.715717154 - 1.027771690i$ and these have absolute values 1.999, 1.999, 2.000. This looks pretty convincing; but how do we actually prove it?

More generally, I'll define a polynomial over $\mathbb{C}$ to be ***good*** if all of its roots have equal absolute value. So how do we tell if a polynomial is good?

2 A Trivial Case: Quadratics

Suppose we're given some quadratic polynomial, which we may as well suppose is monic, x^2+ax+b. Then familiar A-level algebra tells us that the roots α, β satisfy $\alpha+\beta=-a$ and $\alpha\beta=b$.

We may as well re-scale everything by $\sqrt{b}$, and get a new polynomial $x^2+a'x+1$ where $a'=a/\sqrt{b}$; this will obviously be good if and only if the last one was. However, in this case it's clear: if the polynomial is good, then α and β must be complex conjugates of each other (since they are inverses of each other and have the same absolute value), so a' must be real and in the interval [–2 , 2], and the converse is clearly also true.

3 The Cubic Case

Let's try and repeat the same argument for cubics. First, we'll use scaling to reduce the problem to the special case where the constant term is 1, as before; so assume that we are given some general cubic polynomial $P(x)=x^3+ax^2+bx+1$.

lemma: *If P is good then we must have* $b = \bar{a}$.

Proof: Let α, β, γ be the roots of *P*. Then the roots of $P(x) = x^3+\bar{a}x^2+\bar{b}x+1$ are { $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ }, and the roots of x^3+ax^2+bx+1 are { $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ }. If *P* is good, then these two sets are the same, so we must have $b = \bar{a}$.

Now, we've reduced the entire problem to considering polynomials of the form $x^3+ax^2+\bar{a}x+1$; and we know that for any polynomial of this type, the inverses of the roots are a permutation of their complex conjugates. Let *S* denote the subset of $\mathbb{C}$ for which this polynomial is good. How can we find a good description of *S*?

Let's take a little time out and think about the question intuitively. A monic polynomial over the complex numbers is uniquely specified by its roots -- hence, 3 complex parameters, or 6 real ones. The subspace corresponding to roots of polynomials of the form $x^3+ax^2+\bar{a}x+1$ only has real dimension 2, though, since we're only allowed to choose the real and imaginary parts of *a*. On the other hand, how many ways can I choose three points α, β, γ on the unit circle, subject to the requirement that $\alpha\beta\gamma = -1$? I can choose the arguments of α and β arbitrarily, and for any choice of these two numbers there is a unique γ that works. So the subset S of the a-plane which works is also 2-dimensional. Also, it is connected, and compact -- that is, closed and bounded -- because I could only choose my α and β from a connected compact set. So we've now changed the question into a geometric one, about describing a certain locus in the plane.

Indeed, if we let $\alpha = e^{i\theta}$ and $\beta = e^{i\phi}$, we are forced to take $\gamma = e^{i(\theta+\phi)}$, and since $\alpha+\beta+\gamma = -a$ it follows that:

lemma: *The set S is exactly the set of points which may be written in the form* $e^{i(\theta+\phi)}-(e^{i\theta}+e^{i\phi})$.

Note that this equals $-e^{i\theta}-2ie^{-i\theta/2}\sin(\phi+\theta/2)$, so the set we're looking for is a union of straight line segments, each of length 4, corresponding to fixing the value of α . We can equally well choose a value of β or of γ , so each point on the set naturally lies on 3 lines.

Clearly the ends of these lines are boundary points; so the boundary of our set at least contains the curve in the plane described by $-e^{i\theta}\pm 2ie^{-i\theta/2}$. With a bit of imagination, one can visualise this as the image of a marked point on a circle of radius 1 rolling around the inside of a circle of radius 3 centred at the origin in the Argand plane, with the point starting at – 3. This locus is a classical curve called a ***deltoid***. Conversely, it's clear that the lines fill in the whole interior of the deltoid; see Figure 1.

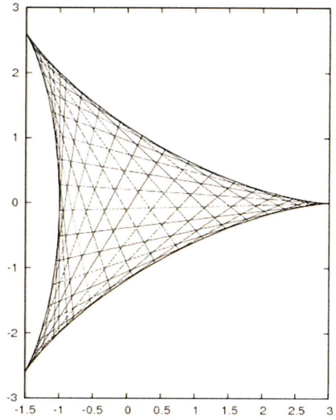


Figure 1: The solution set S for cubic polynomials

Note that every point inside the set is on three lines, and each point on a line corresponds to two values of θ; this gives the six-fold symmetry, corresponding to the 6 ways of ordering the roots of the cubic. The diagram has another sort of six-fold symmetry in that if a is possible, then $\bar{a}$ is, and so is λa for λ each of the three cube roots of 1; so reflection in the x-axis and rotations by $2\pi/3$ preserve it. Points on the edges correspond to cubics with double roots, and the cusps correspond to the three values of a for which $x^3+ax^2+\bar{a}x+1$ has a triple root (the three cube roots of –27).

4 An Algebraic Formula

This picture is very pretty, but it does not completely solve the problem: how do we verify whether or not a given value of a lies inside the deltoid? Here we use the fact that the boundary of S corresponds to good polynomials with a double root, and hence discriminant zero.

The discriminant of $T^3+(x+iy)T^2+(x-iy)T+1$ is easily checked to be $C(x,y)=(x^2+y^2)^2-8x(x^2-3y^2)+18(x^2+y^2)-27$; and it's easily seen that the set S corresponds to $C(x,y)\leq 0$, and the rest of the Argand plane to $C(x,y)\geq 0$.

Let's verify this for my pet example above. We started with the cubic

$$x^3-\frac{-7+\sqrt{-259}}{8}x^2+\frac{-7-\sqrt{-259}}{4}x-8$$

Scaling x by a factor of – 2, we get a polynomial of the form above, $x^3+ax^2+\bar{a}x+1$, for

$$a=\frac{-7+\sqrt{-259}}{16}.$$

This corresponds to the point $(x, y) = (-7/16, \sqrt{259}/16)$ in the (x, y) plane. We find that

$$C(-7/16, \sqrt{259}/16) = -\frac{56727}{4096}$$

so, as required, the original polynomial was good, with three roots of equal absolute value 2.

5 Quartics and Beyond

For a general monic polynomial

$$P(x) = x^d + \sum_{i=1}^{d} a_i x^{d-i}$$

of degree *d*, we can certainly still scale so that $a_d = 1$, and the argument of Lemma 1 implies that if *P* is good, the coefficients must satisfy $a_{d-i} = \bar{a}_i$. In particular, if *d* is even then $a_{d/2}$ is real. So the set *S* naturally lives in $\mathbb{R}^{n-1}$, and as before it is connected, compact, and has dimension $d - 1$.

The projection of *S* onto $\mathbb{R}^2$ that corresponds to the real and imaginary parts of a_1 will be the obvious generalisation of the deltoid, a hypocycloid of *d* cusps. Seeing the effect of the further coordinates is more difficult; a 2-D projection of the 3-dimensional solution set for quartic polynomials is shown in Figure 2 below.

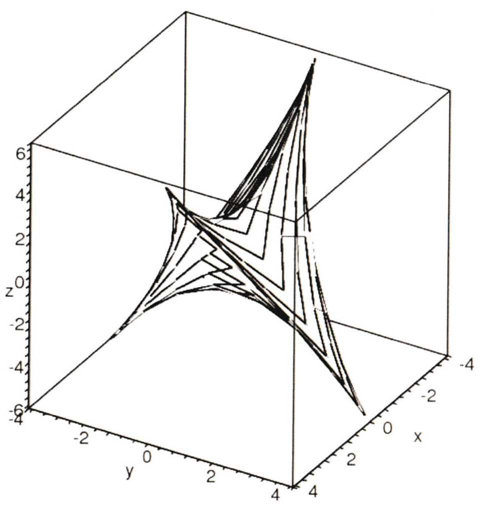


Fig 2. The solution set for quartics: *x* and *y* axes are the real and imaginary parts of the complex number a_1 , and *z* axis is the real number a_2.

The Archimedeans 2007

Julia Erhard, Fitzwilliam College

Once again the society had a very successful year, starting off with about 100 new members, who joined the Archimedeans in freshers' week. Unfortunatly very few first year's got involved in the new committee and plenum, that was appointed in Easter term. Everyone, who is interested in helping to run the society, is more than welcome, since we need keen young mathematicians to get involved, to guarantee that the society can exist and have a very successful year in 2008, too.

The speakers in Michaelmas term were Prof. Michael Duff, who talked about physical applications of Caylay's hyperdeterminante, followed by Prof. Andrew Thomason about whether "threes are more even than odd". The sequence of evening talks was completed traditionally by an comedy show of the impovised comedy ents (ICE). The sequence of talks in Lent term was started off by Prof. Fernando Quevedo with the title "The string landscape and our universe", followed by an interesting survey about Soduko and Latin squares by Prof. Peter Cameron, followed by a talk by Prof. Frances Kirwan about "surfaces and strings", and finally another a comedy show by ICE.

In addition to the evening talks, there were five seminar for undergraduates happening this year. These seminars are aimed to give undergraduates an insight into cutting edge mathematics and what it is like to do research, e.g. this year it was about symplectic topology, knots, category theory or how to use string theory to explore Quantum chromodynamics.
The first one was given by Jonathan Evans, the second by Linda Uruchurtu-Gomez in Michaelmas; then one by Julia Goedecke and the other one by Jack Waldron in Lent; and we even had a SU2 in Easter Term given by the magician Norman Gilbreath, who came all the way from California to give an entertaining presentation and explanation of mathematical card tricks. The bookshop was running all year and has become quite popular, too.

On the social side, there also were lots of events. The Christmas Dinner, a cocktail party and the garden party in may week were jointly held with the other science societies and turned out to be a great success, even if the rain interferred with our plans. Also the weather on the day of the punt trip could have been nicer, but nevertheless a great number of mathematicians couldn't be deterred and punted to the Green Man Pub in Grantchester and back in the darkness. The annual problems drive was competed against the maths societies of Warwick and Oxford and every week there was a puzzles and games ring in Burrell's field in Trinity.

I am now looking forward to the next academic year, which will hopefully be equally successful. We will begin it with this new edition of Eureka, and are going to have the triennial dinner and lots of talks and social events. My thanks goes to all the dedicated people in the committee and plenum, as well as to all subscribers and members of the society, without them I couldn't have started off this article with the same first sentence.

Once Upon A Time

,The winner of the 2007 best mathematical essay competition of the puzzle hunt

Once upon a time there was a bear named Teddy who lived on a roof.

Most of the time Teddy drank coffee out of doughnuts and ate 4-D honey.

But every year, on the evening of the 13th of March, Teddy would gather his reindeer $\{\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots\}$ and head off to Smooth Manifold carrying Klein Bottles filled with $\pi\, s$.

At the stroke of midnight Teddy sent s down the chimneys of each of the buildings in Smooth Manifold: Alpha, Beta, $\mathbb{C}$, Delta, Epsilon, F_p, Gamma and Hamiltonian, in preparation for the celebrations of the next day.

Everyone in Smooth Manifold would gather together to play SET, Settlers over Catalan, Ricochet Robots, and singing Finite Simple Group (of order two).

The celebrations peaked at 1:59 with a π -eating contest, and lasted until 2:71 am. When all the was gone, the principal logarithm of the primitive sixteenth root of unity turned to i and said "I over ate".

Book Reviews

A First Course in Mathematical Analysis by David Brannan, CUP.

Anna Hall, St. John's College

I would recommend 'A First Course in Mathematical Analysis' to students who are finding Analysis difficult, to sixth form students applying for mathematics who are particularly interested in learning Analysis or to anyone who does not have the benefit of good lectures. Why is this book better than any other book on Analysis? Because it has lots of exercises, many with full solutions, as well as large margins.

The large margins in this book are useful for a number of purposes. They contain notes warning of difficult proofs which can be missed at a first reading, these ensure that beginners are not scared off unnecessarily, there are also references to other similar or relevant proofs. Diagrams which are not central to an argument, but which are useful illustrations are put in the margins, this avoids upsetting the flow of an argument. Most importantly the margin is a place where readers can write their own notes, linking the arguments to similar ones in other branches of maths or to questions they are trying to solve.

Brannan's book starts by introducing real numbers in terms of (sometimes non-terminating) decimals. If ε is causing confusion students can become put off Analysis, this book prevents that. By the time the concept is introduced you are familiar with many important ideas including the 'Least Upper Bound Property'. Throughout the book difficult concepts are presented in the same gentle way. The layout of the book ensures that by the time a difficult proof is reached a student will have done plenty of examples and have an intuitive sense of why the result holds.

Although 'A First Course in Mathematical Analysis' has an unhurried pace it eventually covers some more advanced results, such as the convergence of power series and a proof of the irrationality of Π. Essentially it is what it claims to be, a first course in analysis. If you were learning Analysis from this book you would not need any other textbooks, as this one is thorough as well as clearly explained.

To conclude, this is a book to learn Analysis from rather than a book to skim through. Brannan's book is most useful to students who work through the exercises and make notes on the material in the margins. If a complete treatment is required by students who will spend time working through 'A First Course in Mathematical Analysis' then it will contain everything necessary to get to grips with the subject. On the other hand, those who are looking for brief proofs of results for revision will find other, smaller books more suited to that purpose.

Classical Mechanics by R. Douglas Gregory

Michaela Freeland, Newnham

Beginning with a brief introduction to vector algebra and progressing through to the calculus of variations, Legendre transforms and phase space methods, the text follows a typical physics/mathematics lecture course in mechanics. This having been said, it falls somewhat awkwardly between introductory and intermediate principles – going beyond, for example, IB Methods in areas (notably Hamilton's principle and conservations/symmetries) whilst not covering tensor algebra to the depth of IA Vector Calculus. As such, it is probably not suitable as the sole textbook for any single subject – rather it would be relevant to a number of courses in the first and, to a lesser extent, second undergraduate years.

Contributing to this is the clear layout – with numerous diagrams, major theorems being highlighted, and a very comprehensive index included. An adequate number of problems are provided in all chapters, for which, helpfully, complete worked solutions are available in a companion volume. The suggested computational projects would provide a useful challenge for beginning programmers, and follow interesting tangents to the main text.

Gregory's emphasis throughout appears to be on relating the mathematics back to physical situations, rather than on the theory itself. This is true both in the text itself and in the pleasingly large numbers of clearly worked examples. The final third of the book is comprised of 'additional topics' serving just this purpose of applying the mathematical insight gained from the preceeding chapters to exploring the physics in more detail. For example, Lagrangian dynamics is applied to a spinning top and gyrocompass. The informal review of further references at the end of the book provides the ideal guide for the reader seeking to explore many of the given subjects in more depth than *Classical Mechanics* itself provides.

Problems Drive 2007

Sean Lip and Onky Leung

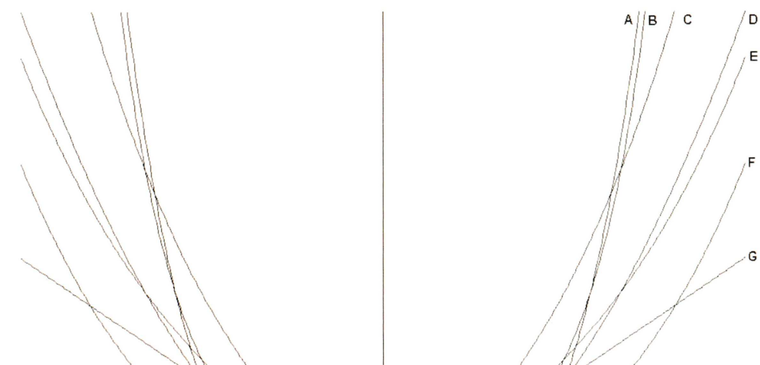
4th March 2007

Question 1

The diagram below is a composite of the following graphs, where the horizontal axis in the diagram is the x-axis:

(1) $1 - e^{-x^2}$	(2) x^2	(3) x^4
(4) $\cosh x - 1$	(5) $\lvert\sinh(x^2)\rvert$	(6) $\lvert\sinh x\rvert$
(7) $e^{\lvert x\rvert} - 1$	(8) $\lvert x\rvert$	

Match each of the above expressions with its corresponding graph.



Question 2

(a) How many pairs (p, n) are there, where p is prime and n is an integer greater than zero, such that $(p^n)!$ has exactly n trailing zeros (in base 10)? If there are fewer than 10, list them all.

[*The number of trailing zeros of a positive integer m is the total number of consecutive zero digits at the end of the number. For example, 38 000 has three trailing zeros, and 20 800 has two.*]

(b) How many pairs (p, n) are there, where p is prime and n is an integer greater than zero, such that $(p^n)!$ has exactly n zeros (in base 10)? If there are fewer than 10, list them all.

Question 3

Between 2003 and 2006, the Euclideans (a University mathematics society) elected six different Presidents. Each President served between one and three consecutive terms in office (the terms in each year are Lent, Easter and Michaelmas, in that order – so the period in question is from Lent 2003 to Michaelmas 2006). Each term is of equal length.

Three of the Presidents were men (Alexander, Gabriel and Robert) and three were women (Emily, Julia and Michaela). Their last names (in alphabetical order) are Bird, Crowston, Freeland, Jones, Shannon and Wu.

Unfortunately, a computer virus has destroyed the Euclideans' archives, but an investigation has uncovered the following details:

1. In consecutive order, three of the Presidents were Shannon, Crowston and Ms Bird.
2. Crowston did not serve in Michaelmas 2004.

3. Emily was elected President immediately after the man who served for just one term.
4. Gabriel and Jones both served before Wu became President.
5. Each female President was succeeded by a man.
6. Julia stepped down from the presidency at the end of Michaelmas 2005.
7. Neither Shannon nor Freeland held office in Lent 2004.
8. The three men together served the society for the same length of time as the three women together.
9. Michaela's last name is not Shannon.
10. Robert's last name is not Jones.

Use the information above to determine the first and last names of each president and when he or she led the Euclideans' committee.

Question 4

In each of the following sequences, determine the two missing terms.

(a) 1, 1, 9, ___, 3, 8, 7, 3, ___, 4, 2, 4, …

(b) ___, 255, 26, 3, 0, …, ___, – 46655/46656, …

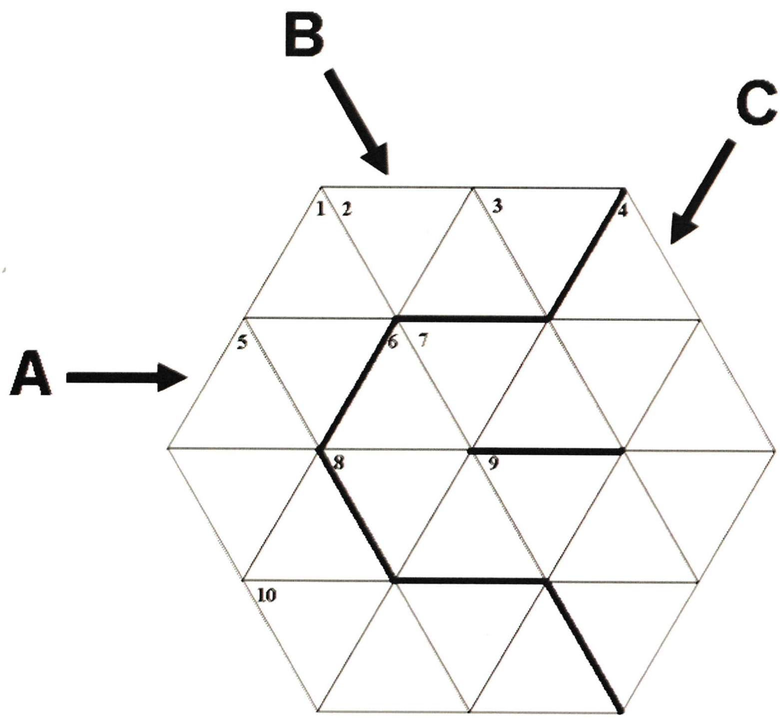
(c) 1, 10, 45, ___, 210, 252, 210, ___, 45, 10, 1

(d) 1, 3, 5, 7, 9, 11, ___, ___, …

(e) 4, 5, 8, 8, 9, 9, 12, 13, 13, ___, ___, …

Question 5

Solve the following cross-number puzzle. Each cell contains one digit, and all answers are positive integers in base 10, with no leading zeroes.



<u>A Clues</u>

1A: A perfect cube.
5A: A perfect square.
6A: A perfect square multiplied by 10.
8A: A perfect cube.
10A: A power of 2.

<u>B Clues</u>

1B: A perfect square.

2B: A prime.

4B: An integer multiple of both **2B** and **3C**.

5B: An integer multiple of **4B** which is palindromic. (A *palindrome* is a number that reads the same backwards as forwards, *e.g.*, 23032, 15551, *etc.*)

6B: Not a prime number.

9B: The product of two different odd primes.

<u>C Clues</u>

3C: One more or one less than an integer multiple of the total number of clues in this Question.

4C: The digits of this number decrease consecutively from left to right. (Examples of such numbers are 210, 543, 98765, *etc.*)

7C: The nth power of a positive integer, for some positive integer n.

Question 6

A rigid ladder is resting against a wall. Initially, it is vertically upright against the wall, and then it starts to slip until it lies horizontally on the floor. The top end of the ladder always remains in contact with the wall.

Sean stands on the midpoint of the ladder. As the ladder moves, his feet trace out part of a simple geometrical figure. Name this figure.

Onky stands two-thirds of the way up the ladder. As the ladder moves, his feet trace out part of a simple geometrical figure. Name this figure.

[*You may treat Sean and Onky as massless point particles for the purposes of this problem.*]

Question 7

In each of the following calculations (in base 10), each letter represents a unique digit from 0 to 9. One of them has no solution; the other has a unique solution. Assign each of the letters in the solvable calculation to its corresponding digit.

```
  MATHS          SIX
+ MATHS        x TWO
 THAMES          DEN
               OZDN
               SIX
               DOZEN
```

Bonus: For an extra point, determine exactly how many solutions there are to the following calculation:

```
  SIX
x TWO
DOZEN
```

Question 8

Determine the exact value of the iterated sum

$$\sum_{i_{2000}=0}^{7} \sum_{i_{1999}=0}^{i_{2000}} \sum_{i_{1998}=0}^{i_{1999}} \cdots \sum_{i_1=0}^{i_2} \sum_{i_0=0}^{i_1} \frac{2000!}{i_0!(2007-i_0)!}.$$

[Hint: You may find it easier to first evaluate

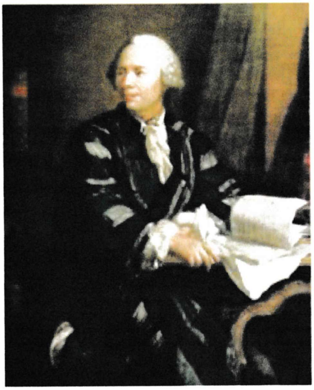
$$\sum_{i_{2000}=0}^{7} \sum_{i_{1999}=0}^{i_{2000}} \sum_{i_{1998}=0}^{i_{1999}} \cdots \sum_{i_1=0}^{i_2} \sum_{i_0=0}^{i_1} f(i_0),$$

where f is some arbitrary function defined on the integers.]

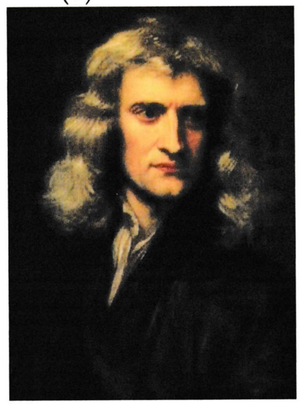
Question 9

What are these people's names?

(a)



(b)



(c)

(d)

(e)

Question 10

Find the exact value of the following integral:

$$\int_1^2 \frac{9x^3 + 16x^2}{x^7 + 3x^4 + 4x^3}\, dx$$

Question 11

In terms of a, determine the volume of a sphere inscribed inside a regular tetrahedron of side length a.

Question 12

Write down an integer between 1 and 1000, inclusive. (*Answers not satisfying this condition will be considered unacceptable and immediately disqualified.*)

The number d is defined to be the absolute value of the difference between your answer and $\lceil \bar{x} \rceil$, where $\bar{x}$ is the mean of all the acceptable answers, and $\lceil \bar{x} \rceil$ denotes the smallest integer greater than or equal to $\bar{x}$.

If $d \leq 9$, your answer will receive $1/(d + 1)$ points. Otherwise, your answer will receive zero points.

Solutions to the Problems Drive 2007

Sean Lip and Onky Leung

4th March 2007

Question 1

This question looks horrendous at first sight, but actually it is not too difficult.

Firstly, it is easy to identify some of the more obvious graphs: (8) corresponds to G, and (1) corresponds to H (since it is the only graph which approaches an asymptote as x tends to infinity). Also, C is the only graph with a cusp at $x = 0$, so it must be (7).

The next thing to note is that there is an obvious point where three graphs (namely, A, D and G) intersect. Noting that (2), (3) and (8) should meet at (1, 1), we can identify (3) with A, and (2) with (D) (since, among other reasons, $x^4 > x^2$ if $x > 1$.)

B and E intersect at $x = 1$, so they must be (5) and (6) in some order; this leaves (4) as F. Since $x^2 > x$ for $x > 1$, we have $\sinh(x^2) > \sinh x$ for $x > 1$. So B is (5), and E is (6).

The answer is therefore: 1H, 2D, 3A, 4F, 5B, 6E, 7C, 8G.

(1/8th point each)

Question 2

(For the first part, notice that the number of trailing zeros in $k!$ is equal to the power of 5 in the prime factorisation of $k!$.) Consider the numbers 2^n, where n is a positive integer. The first few numbers are 2, 4, 8, 16, 32, … which have 0, 0, 1, 3, 7, … trailing zeros respectively. Thereafter, the number of trailing zeros increases much faster than n does, so we reject the case $p = 2$.

We can continue in this manner for $p = 3, 5, 7, 11, \ldots$ – obtaining the possibilities $5^1!$ and $7^1!$, which each have 1 trailing zero. By the time we get to $p = 11$, we notice that $(11^1)!$ has already got 2 trailing zeros, so we can stop here. Thus, there are exactly two solutions: $(p, n) = (5, 1)$ or $(7, 1)$.

(1/6 point for 'two'; 1/6 point for each correct solution but deduct 1/6 point for each incorrect solution, up to a minimum of 0 for this part of the question)

Similarly, for the second part, we only need to consider cases where the number is below that of the relevant bound give in the second table. The only possible candidates are $p^n = 2^1, 2^2, 2^3, 2^4, 3^1, 3^2, 5^1, 7^1$. By checking each in turn, we get exactly two possibilities: $(p, n) = (2, 4)$ or $(5, 1)$.

(1/6 point for 'two'; 1/6 point for each correct solution but deduct 1/6 point for each incorrect solution, up to a minimum of 0 for this part of the question)

Question 3

(In this solution, clue numbers are underlined.)

From 5, and using the fact that there are three Presidents of each gender, we must have the sequence W-M-W-M-W-M (where W = woman, M = man). From 8 and 11, the possible lengths of terms for Presidents of each gender are 1, 2 and 3; or 2, 2 and 2.

Using 6 we find that Julia served in Michaelmas 2005, and was followed by a man who served 3 terms. So one of the other two men must have served for one term, and the other for two terms.

By 3 we find that Emily is the second female President, and is immediately preceded by a man who served one term. So Michaela is the first woman. Then use 9, and 1, and then it is easy to finish off.

The solution is:

Michaela Freeland served from Lent 03 – Michaelmas 03
Alexander Jones served from Lent 04 – Lent 04
Emily Shannon served from Easter 04 – Michaelmas 04
Gabriel Crowston served from Lent 05 – Easter 05
Julia Bird served from Michaelmas 05 – Michaelmas 05
Robert Wu served from Lent 06 – Michaelmas 06

(1/18 point for each correct correspondence. Note that it is all right to give the start and end dates of the President's term, or the start date and length of the term.)

Question 4

(a) These are the odd-numbered digits after the decimal point in the expansion of π = 3.**1**4**1**5**9**2**6**5**3**5**8**9**7**9**3**2**3**8**4**6**2**6**4**3**3**8... The missing terms are 6 and 3.

(b) This sequence consists of numbers of the form $n^n - 1$, where n is an integer, arranged in decreasing order of n. The missing terms are 3124 and –3126/3125.

(c) These are the binomial coefficients ${}^{10}C_k$, for k increasing from 0 to 10 (or, if you like, the eleventh row in Pascal's Triangle). The missing terms are 120 and 120.

(d) These are the positive odd numbers. The missing terms are 13 and 15.

(e) The n^{th} term in this sequence is equal to n + (number of letters in n when written as an English word). Thus the first term is 1 + 3 = 4 (since 'ONE' has three letters). The answers are 13 (= 10 + 'TEN') and 17 (= 11 + 'ELEVEN'). [*No one solved this.*]

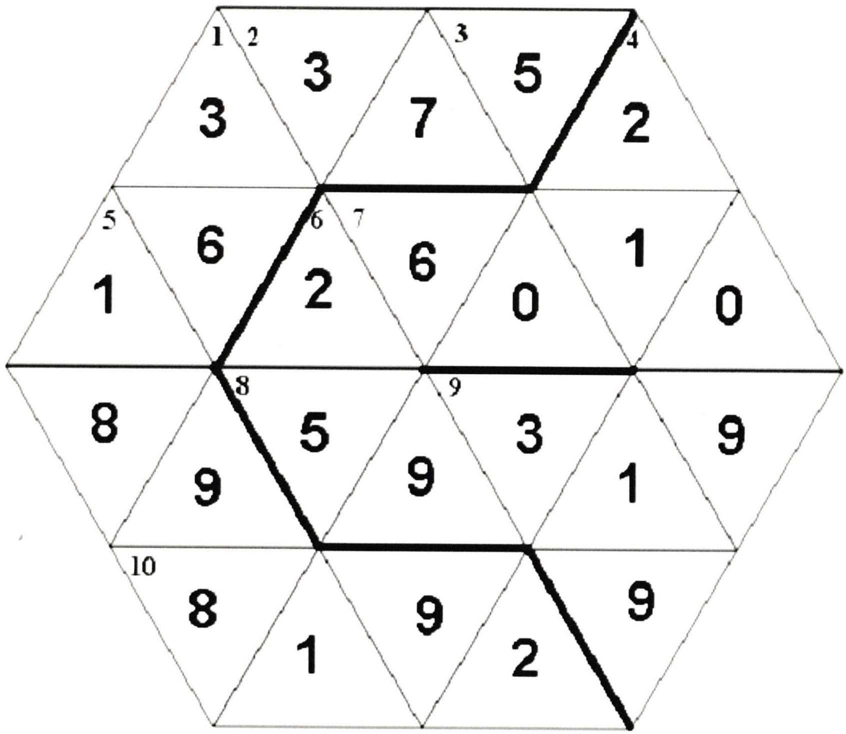
(1/10th point per correct answer)

Question 5

Probably the hardest part of this question is knowing where to start. In this case, the best place is probably the triple of clues **1A**, **2B** and **3C** – consideration of these gives **1A** = 3375. **6A** and **4B** then show that **4B** = 2109. **4C** and **6A** then show that **6A** is 24010, 26010 or 98010.

Using **7C** we then find that **6A** must be 26010, and **7C** is 625. (Note: there is no need to find the prime factorisation of all the numbers from 420 to 429, 620 to 629, *etc.* – it is easier to consider powers of 2, powers of 3, *etc.*, since these rapidly become quite large after just the second power.) Then we can get **8A** = 59319, and **9B** = 319. We also have **1B** = 36.

Finally, **5A** is either 16 or 36, with **5B** = 18981 or 35853 respectively. However, only the former solution works, as there is no four-digit power of 2 starting with 5. Hence, **5B** = 18981, **5A** = 16 and **10A** = 8192.



(1/24 point per correct digit)

Question 6

Put the origin at the intersection of the floor and the wall, with the positive x-axis along the floor and pointing away from the wall. Represent the ladder as a straight line making angle θ with the floor.

For any given θ, the top of the ladder lies at $(0, \sin\theta)$ and the base of the ladder lies at $(\cos\theta, 0)$. Sean's position is given by $(x, y) = ((\cos\theta)/2, (\sin\theta)/2)$, so he traces out the curve $x^2 + y^2 = 1/4$, which is a circle centred at the origin with radius ½. Onky's position is given by $(x, y) = ((\cos\theta)/3, (2\sin\theta)/3)$, so he traces out the curve $x^2/(1/3)^2 + y^2/(2/3)^2 = 1$, which is the equation of an ellipse. *(1/2 point each)*

[*One answer to this question won the 'silliest answer' prize for equating people/point particles and geometric shapes, claiming that "Sean is a circle, Onky is an ellipse."*]

Question 7

The first sum is impossible to solve. If it was solvable, then T would have to be 1, and S would have to be 0. Then M would be 2 or 3 – but then, the sum would be less than 100 000, which is a contradiction.

The second equation is solvable. Clearly T = 1. Also, since E + N = E, N is 0. So X and O are 2 and 5, in some order. If O = 5, then, since SIX times 5 is a three-digit number, S must be 1, but this is impossible since we have ten different letters representing ten different digits, and T is already 1. So X = 5, and O = 2. Then S must be 3 or 4, and, from the leftmost column, 3 + S = D since there must be a carry-over of 1 (as Z and I in the next column are each greater than 2). This gives us two possibilities which we can try out, and we easily get the solution: 345 times 182 = 62790, i.e. S-3, I-4, X-5, T-1, W-8, O-2, D-6, Z-7, E-9, N-0. (*1/10th point per correct match*)

(*Note: no points for stating which of the sums is solvable.*)

Bonus: There are six solutions, namely (345)(182) = 62790, (128)(745) = 95360, (315)(276) = 86940, (156)(307) = 47892, (147)(609) = 89523, (109)(572) = 62348. (*1 point*)

Question 8

Let's use the hint, but generalise even further, replacing 7 and 2000 with m and n respectively. We want to compute:

$$\sum_{i_n=0}^{m} \sum_{i_{n-1}=0}^{i_n} \cdots \sum_{i_1=0}^{i_2} \left(\sum_{i_0=0}^{i_1} f(i_0) \right).$$

Consider the number of terms in the summation when it is written out explicitly. It is clear that this is exactly equal to the number of $(i_0, i_1, \ldots, i_n)$ satisfying $0 \le i_0 \le i_1 \le \ldots \le i_n \le m$, which is ${}^{m+n+1}C_{n+1}$. Furthermore, the number of summands with $i_0 = 0$ is the number of $(i_1, i_2, \ldots, i_n)$ satisfying $0 \le i_1 \le \ldots \le i_n \le m$, which is ${}^{m+n}C_n$; the number of summands with $i_0 = 1$ is the number of $(i_1, i_2, \ldots, i_n)$ satisfying $1 \le i_1 \le \ldots \le i_n \le m$, which is ${}^{m+n-1}C_n$; and, in general, the number of summands with $i_0 = k$, where $0 \le k \le m$, is ${}^{m+n-k}C_n$.

(As a check, we have that ${}^{m+n+1}C_{n+1} = {}^{m+n}C_n + {}^{m+n-1}C_n + \ldots + {}^{n}C_n$: this is called the Hockey Stick Identity (so named because the corresponding numbers in Pascal's Triangle are in the shape of a hockey stick). To prove the identity, consider the set $\{1, 2, \ldots, m + n + 1\}$ and suppose we want to choose $n + 1$ distinct numbers from it. Clearly the smallest of these must be one of 1, 2, …, m; each term on the RHS counts the number of ways to choose the remaining n numbers in each case.)

Thus, the sum reduces to $\sum_{k=0}^{m} \binom{m+n-k}{n} f(k)$, and substituting n = 2000, $m = 7$ and $f(k) = 2000!/k!(2007 - k)!$ gives the answer :

$$\sum_{k=0}^{7} \frac{1}{k!(7-k)!} = \frac{1}{7!}\sum_{k=0}^{7}\binom{7}{k} 1^k 1^{7-k} = \frac{1}{5040}(1+1)^7 = \frac{8}{315}.$$

(1 point for correct answer; $2^7/7!$ also gets the full point. ½ point for reaching $\sum_{k=0}^{7} \frac{1}{k!(7-k)!}$*.)*

Question 9

(a) Leonhard Euler
(b) Isaac Newton
(c) Albert Einstein
[No one got (c) *– submissions received included Turing and Dirac.]*
(d) Kurt Gödel
(e) Carl Friedrich Gauss

(*surnames sufficient; 1/5 point each*)

Question 10

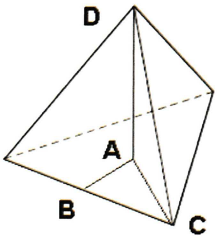
We can write $9x^3 + 16x^2 = (7x^6 + 21x^3 + 28x^2) - (7x^6 + 12x^3 + 12x^2)$. The integral then becomes

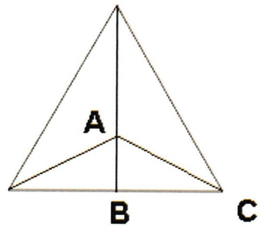
$$\int_1^2 \frac{7}{x}\left(\frac{x^6 + 3x^3 + 4x^2}{x^6 + 3x^3 + 4x^2}\right) dx - \int_1^2 \frac{7x^6 + 12x^3 + 12x^2}{x^7 + 3x^4 + 4x^3}\, dx$$

Notice that the second term can be integrated using the substitution $u = x^7 + 3x^4 + 4x^3$. This gives the answer: 7 log 2 – (log 208 – log 8) = log(64/13) . (*1 point*)

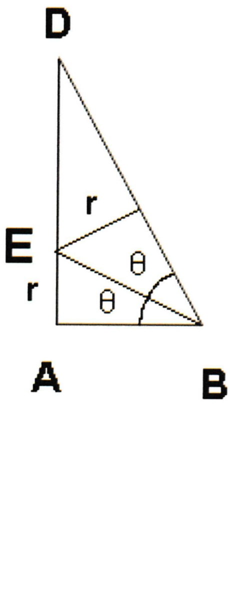
[*No one solved this – but, surprisingly, many answers of '1' were received.*]

Question 11





In the above diagram, A is the centroid of the triangular base, and $BC = a/2$. Angle ACB is 30°, since the base is an equilateral triangle. So $AB = \dfrac{a}{2\sqrt{3}}$ and $AC = \dfrac{a}{\sqrt{3}}$. CD has length a, and AC is known, so $AD^2 = CD^2 - AC^2 = 2a^2/3$.



Now $\tan(2\theta) = AD/AB = 2\sqrt{2}$. Let $t = \tan\theta$.

Using $\tan(2\theta) = 2t/(1 - t^2)$, we obtain $t = \dfrac{1}{\sqrt{2}}$.

Thus $\tan\theta = \dfrac{1}{\sqrt{2}} = \dfrac{r}{AB}$, so $r = \dfrac{a}{2\sqrt{6}}$, and the volume is $\dfrac{\pi\sqrt{6}}{216}a^3$.

Question 12

Basically, this question asks you to write down a positive integer. The average of all the valid answers is calculated, and then rounded up to the nearest integer. The closer you are to the average, the more points you score.

There are interesting cases to be made for writing a number as large as possible, or a number like 990 (to leave room for error on both sides), or something in the middle like 500.

[*Answers received: 4, 100, 500, 501, 501, 501, 501, 534 for an average of 392.75 – which meant that no one got any points for this question! In hindsight, perhaps the range of numbers should have been restricted from just 1 to 100.*]

(1/(d + 1) points for a close enough answer, where d is defined as in the question)

Results

The scores ranged from $2\frac{193}{360}$ to $6\frac{11}{60}$, but the highest score belonged to a team containing at least one previous winner – so **Shegnik Das** and **Ling Zhu** from Fitzwilliam, with the next highest score of $5\frac{1}{5}$, win the bottle of port! Shegnik and Ling will be setting the 2008 Problems Drive.

For lovers of statistics (or those with too much time on their hands), the table below shows the mean number of points awarded per problem:

Problem	Mean number of points scored (out of 1)
1	0.734
2	0.646
3	0.139
4	0.288
5	0.130
6	0.750
7	0.100
8	0.250
9	0.525
10	0.000
11	0.250
12	0.000

A few words from the department

“If (ii) holds with equality we would have an integral. And if pigs could fly we wouldn't need vans to transport them.”

on the existence of a Haar measure

“You should ask your Director of Studies about this. Unless your Director of Studies happens to be me, in which case, you should ask someone else's Director of Studies.”

Algebra and Geometry

“'If your picture of Cambridge is based on `Brideshead Revisited', `A Yank At Oxford' and `Charlie's Aunt' you may be slightly disappointed. One reason is that the works referred to all take place in Oxford...”

On what Cambridge isn't

“Of course we use the famous Swiss Cheese algebra, or as the French call it, the Sweeees Cheeese algebra”

Introduction to Functional Analysis

"I'm sure this is a mistake, if only because I make one mistake each lecture and there's only ten minutes left"

Markov Chains

“That's for the pedants in the audience: a pedant is anyone who disagrees with me.”

On pedantry