Discrete Mathematics

Dr. J. Saxl

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Introduction

These notes are based on the course "Discrete Mathematics" given by Dr. J. Saxl in Cambridge in the Michælmas Term 1995. These typeset notes are totally unconnected with Dr. Saxl.

Other sets of notes are available for different courses. At the time of typing these courses were:

Probability	Discrete Mathematics
Analysis	Further Analysis
Methods	Quantum Mechanics
Fluid Dynamics 1	Quadratic Mathematics
Geometry	Dynamics of D.E.'s
Foundations of QM	Electrodynamics
Methods of Math. Phys	Fluid Dynamics 2
Waves (etc.)	Statistical Physics
General Relativity	Dynamical Systems
Combinatorics	Bifurcations in Nonlinear Convection

They may be downloaded from

`http://www.istari.ucam.org/maths/`.

Chapter 1

Integers

Notation. *The "natural numbers", which we will denote by $\mathbb{N}$, are*

$$\{1, 2, 3, \dots\}.$$

The integers $\mathbb{Z}$ are

$$\{\dots, -2, -1, 0, 1, 2, \dots\}.$$

We will also use the non-negative integers, denoted either by $\mathbb{N}_0$ or $\mathbb{Z}_+$, which is $\mathbb{N} \cup \{0\}$. There are also the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$.

Given a set S, we write $x \in S$ if x belongs to S, and $x \notin S$ otherwise.

There are operations $+$ and $\cdot$ on $\mathbb{Z}$. They have certain "nice" properties which we will take for granted. There is also "ordering". $\mathbb{N}$ is said to be "well-ordered", which means that every non-empty subset of $\mathbb{N}$ has a least element. The principle of induction follows from well-ordering.

Proposition (Principle of Induction)**.** *Let $P(n)$ be a statement about n for each $n \in \mathbb{N}$. Suppose $P(1)$ is true and $P(k)$ true implies that $P(k+1)$ is true for each $k \in \mathbb{N}$. Then P is true for all n.*

Proof. Suppose P is not true for all n. Then consider the subset S of $\mathbb{N}$ of all numbers k for which P is false. Then S has a least element l. We know that $P(l-1)$ is true (since $l > 1$), so that $P(l)$ must also be true. This is a contradiction and P holds for all n. □

1.1 Division

Given two integers a, $b \in \mathbb{Z}$, we say that a divides b (and write $a \mid b$) if $a \neq 0$ and $b = a \cdot q$ for some $q \in \mathbb{Z}$ (a is a divisor of b). a is a *proper divisor* of b if a is not ± 1 or $\pm b$.

Note. *If $a \mid b$ and $b \mid c$ then $a \mid c$, for if $b = q_1 a$ and $c = q_2 b$ for q_1, $q_2 \in \mathbb{Z}$ then $c = (q_1 \cdot q_2)a$. If $d \mid a$ and $d \mid b$ then $d \mid ax + by$. The proof of this is left as an exercise.*

1.2 The division algorithm

Lemma 1.1. *Given a, $b \in \mathbb{N}$ there exist unique integers q, $r \in \mathbb{N}$ with $a = qb + r$, $0 \leq r < b$.*

Proof. Take q the largest possible such that $qb \leq a$ and put $r = a - qb$. Then $0 \leq r < b$ since $a - qb \geq 0$ but $(q+1)b \geq a$. Now suppose that $a = q_1 b + r$ with $q_1, r_1 \in \mathbb{N}$ and $0 \leq r_1 < b$. Then $0 = (q - q_1)b + (r - r_1)$ and $b \mid r - r_1$. But $-b < r - r_1 < b$ so that $r = r_1$ and hence $q = q_1$. $\square$

It is clear that $b \mid a$ iff $r = 0$ in the above.

Definition. *Given a, $b \in \mathbb{N}$ then $d \in \mathbb{N}$ is the highest common factor (greatest common divisor) of a and b if:*

1. *$d \mid a$ and $d \mid b$,*
2. *if $d' \mid a$ and $d' \mid b$ then $d' \mid d$ ($d' \in \mathbb{N}$).*

The highest common factor (henceforth hcf) of a and b is written (a, b) or $\mathrm{hcf}(a, b)$.

The hcf is obviously unique — if c and c' are both hcf's then they both divide each other and are therefore equal.

Theorem 1.1 (Existance of hcf)**.** *For a, $b \in \mathbb{N}$ $\mathrm{hcf}(a, b)$ exists. Moreover there exist integers x and y such that $(a, b) = ax + by$.*

Proof. Consider the set $I = \{ax + by : x, y \in \mathbb{Z} \text{ and } ax + by > 0\}$. Then $I \neq \emptyset$ so let d be the least member of I. Now $\exists x_0, y_0$ such that $d = ax_0 + by_0$, so that if $d' \mid a$ and $d' \mid b$ then $d' \mid d$.

Now write $a = qd + r$ with q, $r \in \mathbb{N}_0$, $0 \leq r < d$. We have $r = a - qd = a(1 - qx_0) + b(-qy_0)$. So $r = 0$, as otherwise $r \in I$: contrary to d minimal. Similiarly, $d \mid b$ and thus d is the hcf of a and b. $\square$

Lemma 1.2. *If a, $b \in \mathbb{N}$ and $a = qb + r$ with q, $r \in \mathbb{N}_0$ and $0 \leq r < b$ then $(a, b) = (b, r)$.*

Proof. If $c \mid a$ and $c \mid b$ then $c \mid r$ and thus $c \mid (b, r)$. In particular, $(a, b) \mid (b, r)$. Now note that if $c \mid b$ and $c \mid r$ then $c \mid a$ and thus $c \mid (a, b)$. Therefore $(b, r) \mid (a, b)$ and hence $(b, r) = (a, b)$. $\square$

1.3 The Euclidean algorithm

Suppose we want to find $(525, 231)$. We use lemmas (1.1) and (1.2) to obtain:

$$
\begin{aligned}
525 &= 2 \times 231 + 63 \\
231 &= 3 \times 63 + 42 \\
63 &= 1 \times 42 + 21 \\
42 &= 2 \times 21 + 0
\end{aligned}
$$

So $(525, 231) = (231, 63) = (63, 42) = (42, 21) = 21$. In general, to find (a, b):

$$
\begin{aligned}
a &= q_1 b + r_1 && \text{with } 0 < r_1 < b \\
b &= q_2 r_1 + r_2 && \text{with } 0 < r_2 < r_1 \\
r_1 &= q_3 r_2 + r_3 && \text{with } 0 < r_3 < r_2 \\
&\vdots \\
r_{i-2} &= q_i r_{i-1} + r_i && \text{with } 0 < r_i < r_{i-1} \\
&\vdots \\
r_{n-3} &= q_{n-1} r_{n-2} + r_{n-1} && \text{with } 0 < r_n < r_{n-1} \\
r_{n-2} &= q_n r_{n-1} + 0.
\end{aligned}
$$

This process must terminate as $b > r_1 > r_2 > \cdots > r_{n-1} > 0$. Using Lemma (1.2), $(a, b) = (b, r_1) = \cdots = (r_{n-2}, r_{n-1}) = r_{n-1}$. So (a, b) is the last non-zero remainder in this process.

We now wish to find x_0 and $y_0 \in \mathbb{Z}$ with $(a, b) = ax_0 + by_0$. We can do this by backsubstitution.

$$
\begin{aligned}
21 &= 63 - 1 \times 42 \\
&= 63 - (231 - 3 \times 63) \\
&= 4 \times 63 - 231 \\
&= 4 \times (525 - 2 \times 231) - 231 \\
&= 4 \times 525 - 9 \times 231.
\end{aligned}
$$

This works in general but can be confusing and wasteful. These numbers can be calculated at the same time as (a, b) if we know we shall need them.

We introduce A_i and B_i. We put $A_{-1} = B_0 = 0$ and $A_0 = B_{-1} = 1$. We iteratively define

$$
\begin{aligned}
A_i &= q_i A_{i-1} + A_{i-2} \\
B_i &= q_i B_{i-1} + B_{i-2}.
\end{aligned}
$$

Now consider $aB_j - bA_j$.

Lemma 1.3.

$$aB_j - bA_j = (-1)^{j+1} r_j.$$

Proof. We shall do this using strong induction. We can easily see that (1.3) holds for $j = 1$ and $j = 2$. Now assume we are at $i \geq 2$ and we have already checked that $r_{i-2} = (-1)^{i-1}(aB_{i-2} - bA_{i-2})$ and $r_{i-i} = (-1)^i (aB_{i-1} - bA_{i-1})$. Now

$$
\begin{aligned}
r_i &= r_{i-2} - q_i r_{i-1} \\
&= (-1)^{i-1}(aB_{i-2} - bA_{i-2}) - q_i (-1)^i (aB_{i-1} - bA_{i-1}) \\
&= (-1)^{i+1}(aB_i - bA_i), \text{ using the definition of } A_i \text{ and } B_i.
\end{aligned}
$$

□

Lemma 1.4.

$$A_i B_{i+1} - A_{i+1} B_i = (-1)^i$$

Proof. This is done by backsubstitution and using the definition of A_i and B_i. □

An immediate corollary of this is that $(A_i, B_i) = 1$.

Lemma 1.5.

$$A_n = \frac{a}{(a,b)} \qquad B_n = \frac{b}{(a,b)}.$$

Proof. (1.3) for $i = n$ gives $aB_n = bA_n$. Therefore $\frac{a}{(a,b)} B_n = \frac{b}{(a,b)} A_n$. Now $\frac{a}{(a,b)}$ and $\frac{b}{(a,b)}$ are coprime. A_n and B_n are coprime and thus this lemma is therefore an immediate consequence of the following theorem. □

Theorem 1.2. *If $d \mid ce$ and $(c, d) = 1$ then $d \mid e$.*

Proof. Since $(c, d) = 1$ we can write $1 = cx + dy$ for some x, $y \in \mathbb{Z}$. Then $e = ecx + edy$ and $d \mid e$. □

Definition. *The least common multiple (lcm) of a and b (written $[a, b]$) is the integer l such that*

1. *$a \mid l$ and $b \mid l$,*
2. *if $a \mid l'$ and $b \mid l'$ then $l \mid l'$.*

It is easy to show that $[a, b] = \frac{ab}{(a,b)}$.

1.4 Applications of the Euclidean algorithm

Take a, b and $c \in \mathbb{Z}$. Suppose we want to find all the solutions $x, y \in \mathbb{Z}$ of $ax + by = c$. A necessary condition for a solution to exist is that $(a, b) \mid c$, so assume this.

Lemma 1.6. *If $(a, b) \mid c$ then $ax + by = c$ has solutions in $\mathbb{Z}$.*

Proof. Take x' and $y' \in \mathbb{Z}$ such that $ax' + by' = (a, b)$. Then if $c = q(a, b)$ then if $x_0 = qx'$ and $y_0 = qy'$, $ax_0 + by_0 = c$. □

Lemma 1.7. *Any other solution is of the form $x = x_0 + \frac{bk}{(a,b)}$, $y = y_0 - \frac{ak}{(a,b)}$ for $k \in \mathbb{Z}$.*

Proof. These certainly work as solutions. Now suppose x_1 and y_1 is also a solution. Then $\frac{a}{(a,b)}(x_0 - x_1) = -\frac{b}{(a,b)}(y_0 - y_1)$. Since $\frac{a}{(a,b)}$ and $\frac{b}{(a,b)}$ are coprime we have $\frac{a}{(a,b)} \mid (y_0 - y_1)$ and $\frac{b}{(a,b)} \mid (x_0 - x_1)$. Say that $y_1 = y_0 - \frac{ak}{(a,b)}$, $k \in \mathbb{Z}$. Then $x_1 = x_0 + \frac{bk}{(a,b)}$. □

1.4.1 Continued Fractions

We return to 525 and 231. Note that

$$\frac{535}{231} = 2 + \frac{63}{231} = 2 + \frac{1}{\frac{231}{63}} = 2 + \frac{1}{3 + \frac{42}{63}} = 2 + \frac{1}{3 + \frac{1}{1+\frac{1}{2}}}.$$

Notation.

$$\frac{535}{231} = 2 + \frac{1}{3+}\frac{1}{1+}\frac{1}{2} = [2, 3, 1, 2] = 2; 3, 1, 2.$$

Note that 2, 3, 1 and 2 are just the q_i's in the Euclidean algorithm. The rational $\frac{a}{b} > 0$ is written as a continued fraction

$$\frac{a}{b} = q_1 + \frac{1}{q_2+}\frac{1}{q_3+}\cdots\frac{1}{q_n},$$

with all the $q_i \in \mathbb{N}_0$, $q_i \geq 1$ for $1 < i < n$ and $q_n \geq 2$.

Lemma 1.8. *Every rational $\frac{a}{b}$ with a and $b \in \mathbb{N}$ has exactly one expression in this form.*

Proof. Existance follows immediately from the Euclidean algorithm. As for uniqueness, suppose that

$$\frac{a}{b} = p_1 + \frac{1}{p_2+}\frac{1}{p_3+}\cdots\frac{1}{p_m}$$

with the p_i's as before. Firstly $p_1 = q_1$ as both are equal to $\lfloor \frac{a}{b} \rfloor$. Since $\frac{1}{p_2+\frac{1}{\cdots}} < 1$ then

$$\left(\frac{a}{b} - p_1\right)^{-1} = p_2 + \frac{1}{p_3 + \frac{1}{\cdots}} = \left(\frac{a}{b} - q_1\right)^{-1} = q_2 + \frac{1}{q_3 + \frac{1}{\cdots}}.$$

Thus $p_2 = q_2$ and so on. $\square$

Now, suppose that given $[q_1, q_2, \ldots, q_n]$ we wish to find $\frac{a}{b}$ equal to it. Then we work out the numbers A_i and B_i as in the Euclidean algorithm. Then $\frac{a}{b} = \frac{A_n}{B_n}$ by lemma (1.3).

If we stop doing this after i steps we get $\frac{A_i}{B_i} = [q_1, q_2, \ldots, q_i]$. The numbers $\frac{A_i}{B_i}$ are called the "convergents" to $\frac{a}{b}$.

Using lemma (1.4), we get that $\frac{A_i}{B_i} - \frac{A_{i-1}}{B_{i-1}} = \frac{(-1)^i}{B_{i-1}B_i}$. Now the B_i are strictly increasing, so the gaps are getting smaller and the signs alternate. We get

$$\frac{A_1}{B_1} < \frac{A_3}{B_3} < \cdots < \frac{a}{b} < \cdots < \frac{A_4}{B_4} < \frac{A_2}{B_2}.$$

The approximations are getting better and better; in fact $\left|\frac{A_i}{B_i} - \frac{a}{b}\right| \leq \frac{1}{B_i B_{i+1}}$.

$*$ — Continued fractions for irrationals

This can also be done for irrationals, but the continued fractions become infinite. For instance we can get approximations to π using the calculator. Take the integral part, print, subtract it, invert and repeat. We get $\pi = [3, 7, 15, 1, \ldots]$. The convergents are 3, $\frac{22}{7}$ and $\frac{333}{106}$. We are already within 10^{-4} of π. There is a good approximation as B_i increases. As an exercise, show that $\sqrt{2} = [1, 2, 2, 2, \ldots]$.

1.5 Complexity of Euclidean Algorithm

Given a and b, how many steps does it take to find (a, b). The Euclidean algorithm is good.

Proposition. *The Euclidean algorithm will find (a, b), $a > b$ in fewer than $5d(b)$ steps, where $d(b)$ is the number of digits of b in base 10.*

Proof. We look at the worst case scenario. What are the smallest numbers needing n steps. In this case $q_i = 1$ for $1 \leq i < n$ and $q_n = 2$. Using these q_i's to calculate A_n and B_n we find the Fibonacci numbers, that is the numbers such that $F_1 = F_2 = 1$, $F_{i+2} = F_{i+1} + F_i$. We get $A_n = F_{n+2}$ and $B_n = F_{n+1}$. So if $b < F_{n+1}$ then fewer than n steps will do. If b has d digits then

$$b \leq 10^d - 1 \leq \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{5d+2} - 1 < F_{5d+2},$$

as

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]. \qquad \text{This will be shown later.}$$

□

1.6 Prime Numbers

A natural number p is a prime iff $p > 1$ and p has no proper divisors.

Theorem 1.3. *Any natural number $n > 1$ is a prime or a product of primes.*

Proof. If n is a prime then we are finished. If n is not prime then $n = n_1 \cdot n_2$ with n_1 and n_2 proper divisors. Repeat with n_1 and n_2. □

Theorem 1.4 (Euclid). *There are infinitely many primes.*

Proof. Assume not. Then let $p_1, p_2, \ldots, p_n$ be all the primes. Form the number $N = p_1 p_2 \ldots p_n + 1$. Now N is not divisible by any of the p_i — but N must either be prime or a product of primes, giving a contradiction. □

This can be made more precise. The following argument of Erdös shows that the k^{th} smallest prime p_k satisfies $p_k \leq 4^{k-1} + 1$. Let M be an integer such that all numbers $\leq M$ can be written as the product of the powers of the first k primes. So any such number can be written

$$m^2 p_1^{i_1} p_2^{i_2} \ldots p_k^{i_k},$$

with $i_1, \ldots, i_k \in \{0, 1\}$. Now $m \leq \sqrt{M}$, so there are at most $\sqrt{M}\, 2^k$ possible numbers less than M. Hence $M \leq 2^k \sqrt{M}$, or $M \leq 4^k$. Hence $p_{k+1} \leq 4^k + 1$.

A much deeper result (which will not be proved in this course!) is the Prime Number Theorem, that $p_k \sim k \log k$.

1.6.1 Uniqueness of prime factorisation

Lemma 1.9. *If $p \mid ab$, $a, b \in \mathbb{N}$ then $p \mid a$ and/or $p \mid b$.*

Proof. If $p \nmid a$ then $(p, a) = 1$ and so $p \mid b$ by theorem (1.2). □

Theorem 1.5. *Every natural number > 1 has a unique expression as the product of primes.*

Proof. The existence part is theorem (1.3). Now suppose $n = p_1 p_2 \dots p_k = q_1 q_2 \dots q_l$ with the p_i's and q_j's primes. Then $p_1 \mid q_1 \dots q_l$, so $p_1 = q_j$ for some j. By renumbering (if necessary) we can assume that $j = 1$. Now repeat with $p_2 \dots p_k$ and $q_2 \dots q_l$, which we know must be equal. □

There are perfectly nice algebraic systems where the decomposition into primes is not unique, for instance $\mathbb{Z}\left[\sqrt{-5}\right] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$, where $6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 2 \times 3$ and $2, 3$ and $1 \pm \sqrt{-5}$ are each "prime". Or alternatively, $2\mathbb{Z} = \{\text{all even numbers}\}$, where "prime" means "not divisible by 4".

1.7 Applications of prime factorisation

Lemma 1.10. *If $n \in \mathbb{N}$ is not a square number then $\sqrt{n}$ is irrational.*

Proof. Suppose $\sqrt{n} = \frac{a}{b}$, with $(a, b) = 1$. Then $nb^2 = a^2$. If $b > 1$ then let p be a prime dividing b. Thus $p \mid a^2$ and so $p \mid a$, which is impossible as $(a, b) = 1$. Thus $b = 1$ and $n = a^2$. □

This lemma can also be stated: "if $n \in N$ with $\sqrt{n} \in \mathbb{Q}$ then $\sqrt{n} \in \mathbb{N}$".

Definition. *A real number θ is algebraic if it satisfies a polynomial equation with coefficients in $\mathbb{Z}$.*

Real numbers which are not algebraic are transcendental (for instance π and e). Most reals are transcendental.

If the rational $\frac{a}{b}$ (with $(a, b) = 1$) satisfies a polynomial with coefficients in $\mathbb{Z}$ then

$$c_n a^n + c_{n-1} a^{n-1} b + \dots b^n c_0 = 0$$

so $b \mid c_n$ and $a \mid c_0$. In particular if $c_n = 1$ then $b = 1$, which is stated as "algebraic integers which are rational are integers".

Note that if $a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ and $b = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$ with $\alpha_i, \beta_i \in \mathbb{N}_0$ then $(a, b) = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_k^{\gamma_k}$ and $[a, b] = p_1^{\delta_1} p_2^{\delta_2} \dots p_k^{\delta_k}$, $\gamma_i = \min\{\alpha_i, \beta_i\}$ and $\delta_i = \max\{\alpha_i, \beta_i\}$.

Major open problems in the area of prime numbers are the Goldbach conjecture ("every even number greater than two is the sum of two primes") and the twin primes conjecture ("there are infinitely many prime pairs p and $p + 2$").

1.8 Modular Arithmetic

Definition. *If a and $b \in \mathbb{Z}$, $m \in \mathbb{N}$ we say that a and b are "congruent mod(ulo) m" if $m \mid a - b$. We write $a \equiv b \pmod{m}$.*

It is a bit like $=$ but less restrictive. It has some nice properties:

- $a \equiv a \pmod{m}$,
- if $a \equiv b \pmod{m}$ then $b \equiv a \pmod{m}$,
- if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $a \equiv c \pmod{m}$.

Also, if $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$

- $a_1 + a_2 \equiv b_1 + b_2 \pmod{m}$,
- $a_1 a_2 \equiv b_1 a_2 \equiv b_1 b_2 \pmod{m}$.

Lemma 1.11. *For a fixed $m \in \mathbb{N}$, each integer is congruent to precisely one of the integers*

$$\{0, 1, \ldots, m-1\}.$$

Proof. Take $a \in \mathbb{Z}$. Then $a = qm + r$ for $q, r \in \mathbb{Z}$ and $0 \leq r < m$. Then $a \equiv r \pmod{m}$. $\square$

If $0 \leq r_1 < r_2 < m$ then $0 < r_2 - r_1 < m$, so $m \nmid r_2 - r_1$ and thus $r_1 \not\equiv r_2 \pmod{m}$.

Example. *No integer congruent to $3 \pmod{4}$ is the sum of two squares.*

Solution. Every integer is congruent to one of $0, 1, 2, 3 \pmod{4}$. The square of any integer is congruent to 0 or $1 \pmod{4}$ and the result is immediate. $\square$

Similarly, using congruence modulo 8, no integer congruent to $7 \pmod{8}$ is the sum of 3 squares.

1.9 Solving Congruences

We wish to solve equations of the form $ax \equiv b \pmod{m}$ given $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$ for $x \in \mathbb{Z}$. We can often simplify these equations, for instance $7x \equiv 3 \pmod{5}$ reduces to $x \equiv 4 \pmod{5}$ (since $21 \equiv 1$ and $9 \equiv 4 \pmod{5}$).

This equations are not always soluble, for instance $6x \equiv 4 \pmod{9}$, as $9 \nmid 6x - 4$ for any $x \in \mathbb{Z}$.

How to do it

The equation $ax \equiv b \pmod{m}$ can have no solutions if $(a, m) \nmid b$ since then $m \nmid ax - b$ for any $x \in \mathbb{Z}$. So assume that $(a, m) \mid b$.

We first consider the case $(a, m) = 1$. Then we can find x_0 and $y_0 \in \mathbb{Z}$ such that $ax_0 + my_0 = b$ (use the Euclidean algorithm to get x' and $y' \in \mathbb{Z}$ such that $ax' + my' = 1$). Then put $x_0 = bx'$ so $ax_0 \equiv b \pmod{m}$. Any other solution is congruent to $x_0 \pmod{m}$, as $m \mid a(x_0 - x_1)$ and $(a, m) = 1$.

So if $(a, m) = 1$ then a solution exists and is unique modulo m.

1.9.1 Systems of congruences

We consider the system of equations

$$\begin{aligned} x &\equiv a \mod m \\ x &\equiv b \mod n. \end{aligned}$$

Our main tool will be the Chinese Remainder Theorem.

Theorem 1.6 (Chinese Remainder Theorem)**.** *Assume $m, n \in \mathbb{N}$ are coprime and let $a, b \in \mathbb{Z}$. Then $\exists x_0$ satisfying simultaneously $x_0 \equiv a \pmod m$ and $x_0 \equiv b \pmod n$. Moreover the solution is unique up to congruence modulo mn.*

Proof. Write $cm + dn = 1$ with $m, n \in \mathbb{Z}$. Then cm is congruent to 0 modulo m and 1 modulo n. Similarly dn is congruent to 1 modulo m and 0 modulo n. Hence $x_0 = adn + bcm$ satifies $x_0 \equiv a \pmod m$ and $x_0 \equiv b \pmod n$. Any other solution x_1 satisfies $x_0 \equiv x_1$ both modulo m and modulo n, so that since $(m, n) = 1$, $mn \mid x_0 - x_1$ and $x_1 \equiv x_0 \pmod{mn}$. ☐

Finally, if $1 < (a, m)$ then replace the congruence with one obtained by dividing by (a, m) — that is consider

$$\frac{a}{(a,m)} x \equiv \frac{b}{(a,m)} \mod \frac{m}{(a,m)}.$$

Theorem 1.7. *If p is a prime then $(p-1)! \equiv -1 \pmod p$.*

Proof. If $a \in \mathbb{N}$, $a \leq p-1$ then $(a, p) = 1$ and there is a unique solution of $ax \equiv 1 \pmod p$ with $x \in \mathbb{N}$ and $x \leq p-1$. x is the inverse of a modulo p. Observe that $a = x$ iff $a^2 \equiv 1 \pmod p$, iff $p \mid (a+1)(a-1)$, which gives that $a = 1$ or $p-1$. Therefore the elements in $\{2, 3, 4, \ldots, p-2\}$ pair off so that $2 \times 3 \times 4 \times \cdots \times (p-2) \equiv 1 \pmod p$ and the theorem is proved. ☐

1.10 Euler's Phi Function

Definition. *For $m \in \mathbb{N}$, define $\phi(m)$ to be the number of nonnegative integers less than m which are coprime to m.*

$\phi(1) = 1$. If p is prime then $\phi(p) = p - 1$ and $\phi(p^a) = p^a\left(1 - \frac{1}{p}\right)$.

Lemma 1.12. *If $m, n \in \mathbb{N}$ with $(m, n) = 1$ then $\phi(mn) = \phi(m)\phi(n)$. ϕ is said to be multiplicative.*

Let $U_m = \{x \in \mathbb{Z} : 0 \leq x < m, (x, m) = 1$, the reduced set of residues or set of invertible elements. Note that $\phi(m) = |U_m|$.

Proof. If $a \in U_m$ and $b \in U_n$ then there exists a unique $x \in U_{mn}$. with $c \equiv a \pmod m$ and $c \equiv b \pmod n$ (by theorem (1.6)). Such a c is prime to mn, since it is prime to m and to n. Conversely, any $c \in U_{mn}$ arises in this way, from the $a \in U_m$ and $b \in U_n$ such that $a \equiv c \pmod m$, $b \equiv c \pmod n$. Thus $|U_{mn}| = |U_m| \, |U_n|$ as required. ☐

An immediate corollary of this is that for any $n \in \mathbb{N}$,

$$\phi(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right).$$

Theorem 1.8 (Fermat-Euler Theorem)**.** *Take* $a,\ m \in \mathbb{N}$ *such that* $(a, m) = 1$. *Then* $a^{\phi(m)} \equiv 1 \pmod m$.

Proof. Multiply each residue r_i by a and reduce modulo m. The $\phi(m)$ numbers thus obtained are prime to m and are all distinct. So the $\phi(m)$ new numbers are just $r_1, \dots, r_{\phi(m)}$ in a different order. Therefore

$$\begin{aligned} r_1 r_2 \dots r_{\phi(m)} &\equiv a r_1 a r_2 \dots a r_{\phi(m)} \pmod m \\ &\equiv a^{\phi(m)} r_1 r_2 \dots r_{\phi(m)} \pmod m. \end{aligned}$$

Since $(m, r_1 r_2 \dots r_{\phi(m)}) = 1$ we can divide to obtain the result. □

Corollary (Fermat's Little Theorem)**.** *If* p *is a prime and* $a \in \mathbb{Z}$ *such that* $p \nmid a$ *then* $a^{p-1} \equiv 1 \pmod p$.

This can also be seen as a consequence of Lagrange's Theorem, since U_m is a group under multiplication modulo m.

Fermat's Little Theorem can be used to check that $n \in \mathbb{N}$ is prime. If $\exists a$ coprime to n such that $a^{n-1} \not\equiv 1 \pmod n$ then n is not prime.

1.10.1 Public Key Cryptography

Private key cryptosystems rely on keeping the encoding key secret. Once it is known the code is not difficult to break. Public key cryptography is different. The encoding keys are public knowledge but decoding remains "impossible" except to legitimate users. It is usually based of the immense difficulty of factorising sufficiently large numbers. At present 150 – 200 digit numbers cannot be factorised in a lifetime.

We will study the RSA system of Rivest, Shamir and Adleson. The user A (for Alice) takes two large primes p_A and q_A with > 100 digits. She obtains $N_A = p_A q_A$ and chooses at random ρ_A such that $(\rho_A, \phi(N_A)) = 1$. We can ensure that $p_A - 1$ and $q_A - 1$ have few factors. Now A publishes the pair N_A and ρ_A.

By some agreed method B (for Bob) codes his message for Alice as a sequence of numbers $M < N_A$. Then B sends A the number $M^{\rho_A} \pmod{N_A}$. When Alice wants to decode the message she chooses d_A such that $d_A \rho_A \equiv 1 \pmod \phi)(N_A)$. Then $M^{\rho_A d_A} \equiv M \pmod{N_A}$ since $M^{\phi(N_A)} \equiv 1$. No-one else can decode messages to Alice since they would need to factorise N_A to obtain $\phi(N_A)$.

If Alice and Bob want to be sure who is sending them messages, then Bob could send Alice $E_A(D_B(M))$ and Alice could apply $E_B D_A$ to get the message — if it's from Bob.

Chapter 2

Induction and Counting

2.1 The Pigeonhole Principle

Proposition (The Pigeonhole Principle)**.** *If $nm + 1$ objects are placed into n boxes then some box contains more than m objects.*

Proof. Assume not. Then each box has at most m objects so the total number of objects is nm — a contradiction. $\square$

A few examples of its use may be helpful.

Example. *In a sequence of at least $kl+1$ distinct numbers there is either an increasing subsequence of length at least $k+1$ or a decreasing subsequence of length at least $l+1$.*

Solution. Let the sequence be $c_1, c_2, \ldots, c_{kl+1}$. For each position let a_i be the length of the longest increasing subsequence starting with c_i. Let d_j be the length of the longest decreasing subsequence starting with c_j. If $a_i \leq k$ and $d_i \leq l$ then there are only at most kl distinct pairs (a_i, d_j). Thus we have $a_r = a_s$ and $d_r = d_s$ for some $1 \leq r < s \leq kl + 1$. This is impossible, for if $c_r < c_s$ then $a_r > a_s$ and if $c_r > c_s$ then $d_r > d_s$. Hence either some $a_i > k$ or $d_j > l$. $\square$

Example. *In a group of 6 people any two are either friends or enemies. Then there are either 3 mutual friends or 3 mutual enemies.*

Solution. Fix a person X. Then X has either 3 friends or 3 enemies. Assume the former. If a couple of friends of X are friends of each other then we have 3 mutual friends. Otherwise, X's 3 friends are mutual enemies. $\square$

Dirichlet used the pigeonhole principle to prove that for any irrational α there are infinitely many rationals $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$.

2.2 Induction

Recall the well-ordering axiom for $\mathbb{N}_0$: that every non-empty subset of $\mathbb{N}_0$ has a least element. This may be stated equivalently as: "there is no infinite descending chain in $\mathbb{N}_0$". We also recall the (weak) principle of induction from before.

Proposition (Principle of Induction). *Let $P(n)$ be a statement about n for each $n \in \mathbb{N}_0$. Suppose $P(k_0)$ is true for some $k_0 \in \mathbb{N}_0$ and $P(k)$ true implies that $P(k+1)$ is true for each $k \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}_0$ such that $n \geq k_0$.*

The favourite example is the Tower of Hanoi. We have n rings of increasing radius and 3 vertical rods (A, B and C) on which the rings fit. The rings are initially stacked in order of size on rod A. The challenge is to move the rings from A to B so that a larger ring is never placed on top of a smaller one.

We write the number of moves required to move n rings as T_n and claim that $T_n = 2^n - 1$ for $n \in \mathbb{N}_0$. We note that $T_0 = 0 = 2^0 - 1$, so the result is true for $n = 0$.

We take $k > 0$ and suppose we have k rings. Now the only way to move the largest ring is to move the other $k-1$ rings onto C (in T_{k-1} moves). We then put the largest ring on rod B (in 1 move) and move the $k-1$ smaller rings on top of it (in T_{k-1} moves again). Assume that $T_{k-1} = 2^{k-1} - 1$. Then $T_k = 2T_{k-1} + 1 = 2^k - 1$. Hence the result is proven by the principle of induction.

2.3 Strong Principle of Mathematical Induction

Proposition (Strong Principle of Induction). *If $P(n)$ is a statement about n for each $n \in \mathbb{N}_0$, $P(k_0)$ is true for some $k_0 \in \mathbb{N}_0$ and the truth of $P(k)$ is implied by the truth of $P(k_0)$, $P(k_0+1)$, $\ldots$, $P(k-1)$ then $P(n)$ is true for all $n \in \mathbb{N}_0$ such that $n \geq k_0$.*

The proof is more or less as before.

Example (Evolutionary Trees). *Every organism can mutate and produce 2 new versions. Then n mutations are required to produce $n+1$ end products.*

Proof. Let $P(n)$ be the statement "n mutations are required to produce $n+1$ end products". P_0 is clear. Consider a tree with $k+1$ end products. The first mutation (the root) produces 2 trees, say with $k_1 + 1$ and $k_2 + 1$ end products with $k_1, k_2 < k$. Then $k + 1 = k_1 + 1 + k_2 + 1$ so $k = k_1 + k_2 + 1$. If both $P(k_1)$ and $P(k_2)$ are true then there are k_1 mutations on the left and k_2 on the right. So in total we have $k_1 + k_2 + 1$ mutations in our tree and $P(k)$ is true is $P(k_1)$ and $P(k_2)$ are true. Hence $P(n)$ is true for all $n \in \mathbb{N}_0$. ☐

2.4 Recursive Definitions

(Or in other words) Defining $f(n)$, a formula or functions, for all $n \in \mathbb{N}_0$ with $n \geq k_0$ by defining $f(k_0)$ and then defining for $k > k_0$, $f(k)$ in terms of $f(k_0)$, $f(k_0+1)$, $\ldots$, $f(k-1)$.

The obvious example is factorials, which can be defined by $n! = n(n-1)!$ for $n \geq 1$ and $0! = 1$.

Proposition. *The number of ways to order a set of n points is $n!$ for all $n \in \mathbb{N}_0$.*

Proof. This is true for $n = 0$. So, to order an n-set, choose the 1st element in n ways and then order the remaining $n-1$-set in $(n-1)!$ ways. ☐

Another example is the Ackermann function, which appears on example sheet 2.

2.5 Selection and Binomial Coefficients

We define a set of polynomials for $m \in \mathbb{N}_0$ as

$$x^{\underline{m}} = x(x-1)(x-2)\dots(x-m+1),$$

which is pronounced "x to the m falling". We can do this recursively by $x^{\underline{0}} = 1$ and $x^{\underline{m}} = (x-m+1)x^{\underline{m-1}}$ for $m > 0$. We also define "x to the m rising" by

$$x^{\overline{m}} = x(x+1)(x+2)\dots(x+m-1).$$

We further define $\binom{x}{m}$ (read "x choose m") by

$$\binom{x}{m} = \frac{x^{\underline{m}}}{m!}.$$

It is also convienient to extend this definition to negative m by $\binom{x}{m} = 0$ if $m < 0$, $m \in \mathbb{Z}$. By fiddling a little, we can see that for $n \in \mathbb{N}$, $n \geq m$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

Proposition. *The number of k-subsets of a given n-set is $\binom{n}{k}$.*

Proof. We can choose the first element to be included in our k-subset in n ways, then then next in $n-1$ ways, down to the k^{th} which can be chosen in $n-k+1$ ways. However, ordering of the k-subset is not important (at the moment), so divide $\frac{n^{\underline{k}}}{k!}$ to get the answer. ☐

Theorem 2.1 (The Binomial Theorem)**.** *For a and $b \in \mathbb{R}$, $n \in \mathbb{N}_0$ then*

$$(a+b)^n = \sum_k \binom{n}{k} a^k b^{n-k}.$$

There are many proofs of this fact. We give one and outline a second.

Proof. $(a+b)^n = (a+b)(a+b)\dots(a+b)$, so the coefficient of $a^k b^{n-k}$ is the number of k-subsets of an n-set — so the coefficient is $\binom{n}{k}$. ☐

Proof. This can also be done by induction on n, using the fact that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

☐

There are a few conseqences of the binomial expansion.

1. For $m, n \in \mathbb{N}_0$ and $n \geq m$, $\binom{n}{m} \in \mathbb{N}_0$ so $m!$ divides the product of any m consecutive integers.

2. Putting $a = b = 1$ in the binomial theorem gives $2^n = \sum_k \binom{n}{k}$ — so the number of subsets of an n-set is 2^n. There are many proofs of this fact. An easy one is by induction on n. Write S_n for the total number of subsets of an n-set. Then $S_0 = 1$ and for $n > 0$, $S_n = 2S_{n-1}$. (Pick a point in the n-set and observe that there are S_{n-1} subsets not containing it and S_{n-1} subsets containing it.

3. $(1-1)^n = 0 = \sum_k \binom{n}{k}(-1)^k$ — so in any finite set the number of subsets of even sizes equals the number of subsets of odd sizes.

It also gives us another proof of Fermat's Little Theorem: if p is prime then $a^p \equiv a \pmod{p}$ for all $a \in \mathbb{N}_0$.

Proof. It is done by induction on a. It is obviously true when $a = 0$, so take $a > 0$ and assume the theorem is true for $a - 1$. Then

$$\begin{aligned} a^p &= ((a-1)+1)^p \\ &\equiv (a-1)^p + 1 \mod p \quad \text{as } \binom{p}{k} \equiv 0 \pmod{p} \text{ unless } k = 0 \text{ or } k = p \\ &\equiv a - 1 + 1 \mod p \\ &\equiv a \mod p \end{aligned}$$

□

2.5.1 Selections

The number of ways of choosing m objects out of n objects is

	ordered	unordered
no repeats	$n^{\underline{m}}$	$\binom{n}{m}$
repeats	n^m	$\binom{n-m+1}{m}$

The only entry that needs justification is $\binom{n-m+1}{m}$. But there is a one-to-one correspondance betwen the set of ways of choosing m out of n unordered with possible repeats and the set of all binary strings of length $n + m - 1$ with m zeros and $n - 1$ ones. For suppose there are m_i occurences of element i, $m_i \geq 0$. Then

$$\sum_{i=1}^{n} m_i = m \leftrightarrow \underbrace{0\dots0}_{m_1}1\underbrace{0\dots0}_{m_2}1\dots1\underbrace{0\dots0}_{m_n}.$$

There are $\binom{n-m+1}{m}$ such strings (choosing where to put the 1's).

2.5.2 Some more identities

Proposition.

$$\binom{n}{k} = \binom{n}{n-k} \qquad n \in \mathbb{N}_0, k \in \mathbb{Z}$$

Proof. For: choosing a k-subset is the same as choosing an $n-k$-subset to reject. □

Proposition.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad n \in \mathbb{N}_0, k \in \mathbb{Z}$$

Proof. This is trivial if $n < 0$ or $k \leq 0$, so assume $n \geq 0$ and $k > 0$. Choose a special element in the n-set. Any k-subset will either contain this special element (there are $\binom{n-1}{k-1}$ such) or not contain it (there are $\binom{n-1}{k}$ such). □

In fact

Proposition.

$$\binom{x}{k} = \binom{x-1}{k-1} + \binom{x-1}{k} \qquad k \in \mathbb{Z}$$

Proof. Trivial if $k < 0$, so let $k \geq 0$. Both sides are polynomials of degree k and are equal on all elements of $\mathbb{N}_0$ and so are equal as polynomials as a consequence of the Fundamental Theorem of Algebra. This is the "polynomial argument". □

This can also be proved from the definition, if you want to.

Proposition.

$$\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k} \quad m, k \in \mathbb{Z}.$$

Proof. If $k < 0$ or $m < k$ then both sides are zero. Assume $m \geq k \geq 0$. Assume $x = n \in \mathbb{N}$ (the general case follows by the polynomial argument). This is "choosing a k-subset contained in an m-subset of a n-set". □

Proposition.

$$\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1} \qquad k \in \mathbb{Z} \setminus \{0\}$$

Proof. We may assume $x = n \in \mathbb{N}$ and $k > 0$. This is "choosing a k-team and its captain". □

Proposition.

$$\binom{n+1}{m+1} = \sum_{k=0}^{n} \binom{k}{m}, \qquad m, n \in \mathbb{N}_0$$

Proof. For

$$\binom{n+1}{m+1} = \binom{n}{m} + \binom{n}{m+1} = \binom{n}{m} + \binom{n-1}{m} + \binom{n-1}{m+1}$$

and so on. □

A consequence of this is that $\sum_{k=1}^{n} k^{\underline{m}} = \frac{1}{m+1}(n+1)^{\underline{m+1}}$, which is obtained by multiplying the previous result by $m!$. This can be used to sum $\sum_{k=1}^{n} k^m$.

Proposition.

$$\binom{r+s}{m+n} = \sum_{k} \binom{r}{m+k}\binom{s}{n-k} \qquad r, s, m, n \in \mathbb{Z}$$

Proof. We can replace n by $m+n$ and k by $m+k$ and so we may assume that $m = 0$. So we have to prove:

$$\binom{r+s}{n} = \sum_{k} \binom{r}{k}\binom{s}{n-k} \qquad r, s, n \in \mathbb{Z}.$$

Take an $(r+s)$-set and split it into an r-set and an s-set. Choosing an n-subset amounts to choosing a k-subset from the r-set and an $(n-k)$-subset from the s-set for various k. □

2.6 Special Sequences of Integers

2.6.1 Stirling numbers of the second kind

Definition. *The Stirling number of the second kind, $S(n,k)$, $n,k \in \mathbb{N}_0$ is defined as the number of partitions of $\{1,\dots,n\}$ into exactly k non-empty subsets. Also $S(n,0)=0$ if $n>0$ and 1 if $n=0$.*

Note that $S(n,k)=0$ if $k>n$, $S(n,n)=1$ for all n, $S(n,n-1)=\binom{n}{2}$ and $S(n,2)=2^{n-1}-1$.

Lemma 2.1. *A recurrence: $S(n,k)=S(n-1,k-1)+kS(n-1,k)$.*

Proof. In any partition of $\{1,\dots,n\}$, the element n is either in a part on its own ($S(n-1,k-1)$ such) or with other things ($kS(n-1,k)$ such). □

Proposition. *For $n \in \mathbb{N}_0$, $x^n = \sum_k S(n,k)x^{\underline{k}}$.*

Proof. Proof is by induction on n. It is clearly true when $n=0$, so take $n>0$ and assume the result is true for $n-1$. Then

$$\begin{aligned}
x^n &= xx^{n-1}\\
&= x\sum_k S(n-1,k)x^{\underline{k}}\\
&= \sum_k S(n-1,k)x^{\underline{k}}(x-k+k)\\
&= \sum_k S(n-1,k)x^{\underline{k+1}} + \sum_k kS(n-1,k)x^{\underline{k}}\\
&= \sum_k S(n-1,k-1)x^{\underline{k}} + \sum_k kS(n-1,k)x^{\underline{k}}\\
&= \sum_k S(n,k)x^{\underline{k}} \text{ as required.}
\end{aligned}$$

□

2.6.2 Generating Functions

Recall the Fibonacci numbers, F_n such that $F_1=F_2=1$ and $F_{n+2}=F_{n+1}+F_n$. Suppose that we wish to obtain a closed formula.

First method

Try a solution of the form $F_n=\alpha^n$. Then we get $\alpha^2-\alpha-1=0$ and $\alpha=\frac{1\pm\sqrt{5}}{2}$. We then take

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

and use the initial conditions to determine A and B. It turns out that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Note that $\frac{1+\sqrt{5}}{2} > 1$ and $\left|\frac{1-\sqrt{5}}{2}\right| < 1$ so the solution grows exponentially. A shorter form is that F_n is the nearest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$.

Second Method

Or we can form an ordinary generating function

$$G(z) = \sum_{n\geq 0} F_n z^n.$$

Then using the recurrence for F_n and initial conditions we get that $G(z)(1-z-z^2) = z$. We wish to find the coefficient of z^n in the expansion of $G(z)$ (which is denoted $[z^n]G(z)$). We use partial fractions and the binomial expansion to obtain the same result as before.

In general, the ordinary generating function associated with the sequence $(a_n)_{n\in\mathbb{N}_0}$ is $G(z) = \sum_{n\geq 0} a_n z^n$, a "formal power series". It is deduced from the recurrence and the initial conditions.

Addition, subtraction, scalar multiplication, differentiation and integration work as expected. The new thing is the "product" of two such series:

$$\sum_{k\geq 0} a_k z^k \sum_{l\geq 0} b_l z^l = \sum_{n\geq 0} c_n z^n, \qquad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

$(c_n)_{n\in\mathbb{N}_0}$ is the "convolution" of the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$. Some functional substitution also works.

Any identities give information about the coefficients. We are not concered about convergence, but within the radius of convergence we get extra information about values.

2.6.3 Catalan numbers

A binary tree is a tree where each vertex has a left child or a right child or both or neither. The Catalan number C_n is the number of binary trees on n vertices.

Lemma 2.2.

$$C_n = \sum_{0\leq k\leq n-1} C_k C_{n-1-k}$$

Proof. On removing the root we get a left subtree of size k and a right subtree of size $n-1-k$ for $0 \leq k \leq n-1$. Summing over k gives the result. □

This looks like a convolution. In fact, it is $[z^{n-1}]C(z)^2$ where

$$C(z) = \sum_{n\geq 0} C_n z^n.$$

We observe that therefore $C(z) = zC(z)^2 + 1$, where the multiplication by z shifts the coefficients up by 1 and then $+1$ adjusts for C_0. This equation can be solved for $C(z)$ to get

$$C(z) = \frac{1 \pm \sqrt{1-4z}}{2z}.$$

Since $C(0) = 1$ we must have the $-$ sign. From the binomial theorem

$$(1-4z)^{\frac{1}{2}} = \sum_{k \geq 0} \binom{\frac{1}{2}}{k} (-4)^k z^k.$$

Thus $C_n = -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1}$. Simplifying this we obtain $C_n = \frac{1}{n+1}\binom{2n}{n}$ and note the corollary that $(n+1) \mid \binom{2n}{n}$.

Other possible definitions for C_n are:

- The number of ways of bracketing $n+1$ variables.
- The number of sequences of length $2n$ with n each of ± 1 such that all partial sums are non-negative.

2.6.4 Bell numbers

Definition. *The Bell number B_n is the number of partitions of $\{1, \ldots, n\}$.*

It is obvious from the definitions that $B_n = \sum_k S(n,k)$.

Lemma 2.3.

$$B_{n+1} = \sum_{0 \leq k \leq n} \binom{n}{k} B_k$$

Proof. For, put the element $n+1$ in with a k-subset of $\{1, \ldots, n\}$ for $k = 0$ to $k = n$. $\square$

There isn't a nice closed formula for B_n, but there is a nice expression for its exponential generating function.

Definition. *The exponential generating function that is associated with the sequence $(a_n)_{n \in \mathbb{N}_0}$ is*

$$\hat{A}(z) = \sum_n \frac{a_n}{n!} z^n.$$

If we have $\hat{A}(z)$ and $\hat{B}(z)$ (with obvious notation) and $\hat{A}(z)\hat{B}(z) = \sum_n \frac{c_n}{n!} z^n$ then $c_n = \sum_k \binom{n}{k} a_k b_{n-k}$, the exponential convolution of $(a_n)_{n \in \mathbb{N}_0}$ and $(b_n)_{n \in \mathbb{N}_0}$.

Hence B_{n+1} is the coefficient of z^n in the exponential convolution of the sequences $1, 1, 1, 1, \ldots$ and $B_0, B_1, B_2, \ldots$. Thus $\hat{B}(z)' = e^z \hat{B}(z)$. (Shifting is achieved by differentiation for exponential generating functions.) Therefore $\hat{B}(z) = e^{e^z + C}$ and using the condition $\hat{B}(0) = 1$ we find that $C = -1$. So

$$\hat{B}(z) = e^{e^z - 1}.$$

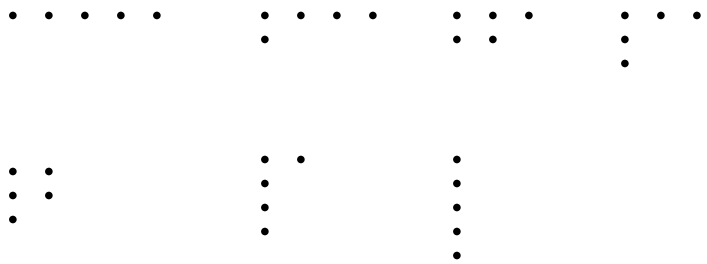
2.6.5 Partitions of numbers and Young diagrams

For $n \in \mathbb{N}$ let $p(n)$ be the number of ways to write n as the sum of natural numbers. We can also define $p(0) = 1$.

For instance, $p(5) = 7$:

	5	$4+1$	$3+2$	$3+1+1$
Notation	5	$4\,1^3$	$3\,2$	$3\,1^2$
	$2+2+1$	$2+1+1+1$	$1+1+1+1+1$	
Notation	$2^2\,1$	$2\,1^3$	1^5	

These partitions of n are usefully pictured by Young diagrams.



The "conjugate partition" is obtained by taking the mirror image in the main diagonal of the Young diagram. (Or in other words, consider columns instead of rows.)

By considering (conjugate) Young diagrams this theorem is immediate.

Theorem 2.2. *The number of partions of n into exactly k parts equals the number of partitions of n with largest part k.*

We now define an ordinary generating function for $p(n)$

$$P(z) = 1 + \sum_{n\in\mathbb{N}} p(n)z^n.$$

Proposition.

$$P(z) = \frac{1}{1-z}\frac{1}{1-z^2}\frac{1}{1-z^3}\cdots = \prod_{k\in\mathbb{N}} \frac{1}{1-z^k}.$$

Proof. The RHS is $(1+z+z^2+\dots)(1+z^2+z^4+\dots)(1+z^3+z^6\dots)\dots$.

We get a term z^n whenever we select z^{a_1} from the first bracket, z^{2a_2} from the second, z^{3a_3} from the third and so on, and $n = a_1 + 2a_2 + 3a_3 + \dots$, or in other words $1^{a_1}\, 2^{a_2}\, 3^{a_3}\, \dots$ is a partition of n. There are $p(n)$ of these. □

We can similarly prove these results.

Proposition. *The generating function $P_m(z)$ of the sequence $p_m(n)$ of partitions of n into at most m parts (or the generating function for the sequence $p_m(n)$ of partitions of n with largest part $\leq m$) satisfies*

$$P_m(z) = \frac{1}{1-z}\frac{1}{1-z^2}\frac{1}{1-z^3}\cdots\frac{1}{1-z^m}.$$

Proposition. *The generating function for the number of partitions into odd parts is*

$$\frac{1}{1-z}\frac{1}{1-z^3}\frac{1}{1-z^5}\cdots.$$

Proposition. *The generating function for the number of partitions into unequal parts is*

$$(1+z)(1+z^2)(1+z^3)\dots.$$

Theorem 2.3. *The number of partitions of n into odd parts equals the number of partitions of n into unequal parts.*

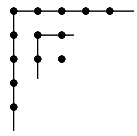
Proof.

$$(1+z)(1+z^2)(1+z^3)\ldots = \frac{1-z^2}{1-z}\frac{1-z^4}{1-z^2}\frac{1-z^6}{1-z^3}\cdots$$
$$= \frac{1}{1-z}\frac{1}{1-z^3}\frac{1}{1-z^5}\cdots$$

□

Theorem 2.4. *The number of self-conjugate partitions of n equals the number of partitions of n into odd unequal parts.*

Proof. Consider hooks along the main diagonal like this.



This process can be reversed, so there is a one-to-one correspondance. □

2.6.6 Generating function for self-conjugate partitions

Observe that any self-conjugate partition consists of a largest $k \times k$ subsquare and twice a partition of $\frac{1}{2}(n-k^2)$ into at most k parts. Now

$$\frac{1}{(1-z^2)(1-z^4)\ldots(1-z^{2m})}$$

is the generating function for partitions of n into even parts of size at most $2m$, or alternatively the generating function for partitions of $\frac{1}{2}n$ into parts of size $\leq m$. We deduce that

$$\frac{z^l}{(1-z^2)(1-z^4)\ldots(1-z^{2m})}$$

is the generating function for partitions of $\frac{1}{2}(n-l)$ into at most m parts. Hence the generating function for self-conjugate partitions is

$$1+\sum_{k\in\mathbb{N}} \frac{z^{k^2}}{(1-z^2)(1-z^4)\ldots(1-z^{2k})}.$$

Note also that this equals

$$\prod_{k\in\mathbb{N}_0}(1+z^{2k+1}),$$

as the number of self-conjugate partitions of n equals the number of partitions of n into unequal odd parts.

In fact in any partition we can consider the largest $k \times k$ subsquare, leaving two partitions of at most k parts, one of $(n-k^2-j)$, the other of j for some j. The number of these two lots are the coefficients of z^{n-k^2-j} and z^j in $\prod_{i=1}^{k}\frac{1}{1-z^i}$ respectively. Thus

$$P(z) = 1+\sum_{k\in\mathbb{N}} \frac{z^{k^2}}{\left((1-z)(1-z^2)\ldots(1-z^k)\right)^2}.$$

Chapter 3

Sets, Functions and Relations

3.1 Sets and indicator functions

We fix some universal set S. We write $P(S)$ for the set of all subsets of S — the "power set" of S. If S is finite with $|S| = m$ (the number of elements), then $|P(S)| = 2^m$.

Given a subset A of S ($A \subseteq S$) we define the "complement" $\bar{A}$ of A in S as $\bar{A} = \{s \in S : s \notin A\}$.

Given two subsets A, B of S we can define various operations to get new subsets of S.

$$\begin{aligned}
A \cap B &= \{s \in S : s \in A \text{ and } s \in B\} \\
A \cup B &= \{s \in S : s \in A \text{ (inclusive) or } s \in B\} \\
A \setminus B &= \{s \in A : s \notin B\} \\
A \circ B &= \{s \in S : s \in A \text{ (exclusive) or } s \in B\} \qquad \text{the symmetric difference} \\
&= (A \cup B) \setminus (A \cap B) \\
&= (A \setminus B) \cup (B \setminus A).
\end{aligned}$$

The indicator function I_A of the subset A of S is the function $I_A \colon S \mapsto \{0, 1\}$ defined by

$$I_A(s) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise.} \end{cases}$$

It is also known as the characteristic function χ_A. Two subsets A and B of S are equal iff $I_A(s) = I_B(s)\ \forall s \in S$. These relations are fairly obvious:

$$\begin{aligned}
I_{\bar{A}} &= 1 - I_A \\
I_{A \cap B} &= I_A \cdot I_B \\
I_{A \cup B} &= I_A + I_B \\
I_{A \circ B} &= I_A + I_B \mod 2.
\end{aligned}$$

Proposition. $A \circ (B \circ C) = (A \circ B) \circ C$.

Proof. For, modulo 2,

$$I_{A \circ (B \circ C)} = I_A + I_{B \circ C} = I_A + I_B + I_C = I_{A \circ B} + I_C = I_{(A \circ B) \circ C} \mod 2.$$

□

Thus $P(S)$ is a group under $\circ$. Checking the group axioms we get:

- Given $A, B \in P(S)$, $A \circ B \in P(S)$ — closure,
- $A \circ (B \circ C) = (A \circ B) \circ C$ — associativity,
- $A \circ \emptyset = A$ for all $A \in P(S)$ — identity,
- $A \circ A = \emptyset$ for all $A \in P(S)$ — inverse.

We note that $A \circ B = B \circ A$ so that this group is abelian.

3.1.1 De Morgan's Laws

Proposition. *1.* $\overline{A \cap B} = \bar{A} \cup \bar{B}$

2. $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof.

$$\begin{aligned} I_{\overline{A \cap B}} &= 1 - I_{A \cap B} = 1 - I_A I_B \\ &= (1 - I_A) + (1 - I_B) - (1 - I_A)(1 - I_B) \\ &= I_{\bar{A}} + I_{\bar{B}} - I_{\bar{A} \cap \bar{B}} \\ &= I_{\bar{A} \cup \bar{B}}. \end{aligned}$$

We prove 2 by using 1 on $\bar{A}$ and $\bar{B}$. □

A more general version of this is: Suppose $A_1, \dots, A_n \subseteq S$. Then

1. $\overline{\bigcap_{i=1}^n A_i} = \bigcup_{i=1}^n \bar{A}_i$
2. $\overline{\bigcup_{i=1}^n A_i} = \bigcap_{i=1}^n \bar{A}_i$.

These can be proved by induction on n.

3.1.2 Inclusion-Exclusion Principle

Note that $|A| = \sum_{s \in S} I_A(s)$.

Theorem 3.1 (Principle of Inclusion-Exclusion)**.** *Given* $A_1, \dots, A_n \subseteq S$ *then*

$$|A_1 \cup \dots \cup A_n| = \sum_{\emptyset \neq J \subseteq \{1, \dots, n\}} (-1)^{|J|-1} |A_J|, \text{where } A_J = \bigcap_{i \in J} A_i.$$

Proof. We consider $\overline{A_1 \cup \dots \cup A_n}$ and note that

$$\begin{aligned} I_{\overline{A_1 \cup \dots \cup A_n}} &= I_{\bar{A_1} \cap \dots \cap \bar{A_n}} \\ &= I_{\bar{A_1}} I_{\bar{A_2}} \dots I_{\bar{A_n}} \\ &= (1 - I_{A_1})(1 - I_{A_2}) \dots (1 - I_{A_n}) \\ &= \sum_{J \subseteq \{1, \dots, n\}} (-1)^{|J|} I_{A_J}, \end{aligned}$$

Summing over $s \in S$ we obtain the result

$$\left|\overline{A_1 \cup \cdots \cup A_n}\right| = \sum_{J \subseteq \{1,\ldots,n\}} (-1)^{|J|} \left|A_J\right|,$$

which is equivalent to the required result. $\square$

Just for the sake of it, we'll prove it again!

Proof. For each $s \in S$ we calculate the contribution. If $s \in S$ but s is in no A_i then there is a contribution 1 to the left. The only contribution to the right is $+1$ when $J = \emptyset$. If $s \in S$ and $K = \{i \in \{1, \ldots, n\} : s \in A_i\}$ is non-empty then the contribution to the right is $\sum_{I \subseteq K} (-1)^{|I|} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i = 0$, the same as on the left. $\square$

Example (Euler's Phi Function)**.**

$$\phi(m) = m \prod_{\substack{p \text{ prime} \\ p|m}} \left(1 - \frac{1}{p}\right).$$

Solution. Let $m = \prod_{i=1}^{n} p_i^{a_i}$, where the p_i are distinct primes and $a_i \in \mathbb{N}$. Let A_i be the set of integers less than m which are divisible by p_i. Hence $\phi(m) = \left|\bigcap_{i=1}^{n} \bar{A}_i\right|$. Now $|A_i| = \frac{m}{p_i}$, in fact for $J \subseteq \{1, \ldots, m\}$ we have $|A_J| = \frac{m}{\prod_{i \in J} p_i}$. Thus

$$\begin{aligned}
\phi(m) &= m - \frac{m}{p_1} - \frac{m}{p_2} - \cdots - \frac{m}{p_n} \\
&+ \frac{m}{p_1 p_2} + \frac{m}{p_1 p_3} + \cdots + \frac{m}{p_2 p_3} + \cdots + \frac{m}{p_{n-1} p_n} \\
&\vdots \\
&+ (-1)^n \frac{m}{p_1 p_2 \ldots p_n} \\
&= m \prod_{\substack{p \text{ prime} \\ p|m}} \left(1 - \frac{1}{p}\right) \qquad \text{as required.}
\end{aligned}$$

$\square$

Example (Derangements)**.** *Suppose we have n psychologists at a meeting. Leaving the meeting they pick up their overcoats at random. In how many ways can this be done so that none of them has his own overcoat. This number is D_n, the number of derangements of n objects.*

Solution. Let A_i be the number of ways in which psychologist i collects his own coat. Then $D_n = \left|\bar{A}_1 \cap \cdots \cap \bar{A}_n\right|$. If $J \subseteq \{1, \ldots, n\}$ with $|J| = k$ then $|A_J| = (n-k)!$. Thus

$$\begin{aligned}
\left|\bar{A}_1 \cap \cdots \cap \bar{A}_n\right| &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \ldots \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\end{aligned}$$

Thus D_n is the nearest integer to $n!\, e^{-1}$, since $\frac{D_n}{n!} \to e^{-1}$ as $n \to \infty$. $\square$

3.2 Functions

Let A, B be sets. A function (or mapping, or map) $f\colon A \mapsto B$ is a way to associate a unique image $f(a) \in B$ with each $a \in A$. If A and B are finite with $|A| = m$ and $|B| = n$ then the set of all functions from A to B is finite with n^m elements.

Definition. *The function $f\colon A \mapsto B$ is injective (or one-to-one) if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$ for all $a_1, a_2 \in A$.*

The number of injective functions from an m-set to an n-set is $n^{\underline{m}}$.

Definition. *The function $f\colon A \mapsto B$ is surjective (or onto) if each $b \in B$ has at least one preimage $a \in A$.*

The number of surjective functions from an m-set to an n-set is $n!\, S(m, n)$.

Definition. *The function $f\colon A \mapsto B$ is bijective if it is both injective and surjective.*

If A and B are finite then $f\colon A \mapsto B$ can only be bijective if $|A| = |B|$. If $|A| = |B| < \infty$ then any injection is a bijection; similarly any surjection is a bijection. There are $n!$ bijections between two n-sets.

If A and B are infinite then there exist injections which are not bijections and *vice versa*. For instance if $A = B = \mathbb{N}$, define

$$f(n) = \begin{cases} 1 & n = 1 \\ n-1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(n) = n + 1.$$

Then f is surjective but not injective and g is injective but not surjective.

Proposition.

$$n!\, S(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$$

Proof. This is another application of the Inclusion-Exclusion principle. Consider the set of functions from A to B with $|A| = m$ and $|B| = n$. For any $i \in B$, define X_i to be the set of functions avoiding i.

So the set of surjections is $\bar{X}_1 \cap \cdots \cap \bar{X}_n$. Thus the number of surjections from A to B is $\left|\bar{X}_1 \cap \cdots \cap \bar{X}_n\right|$. By the inclusion-exclusion principle this is $\sum_{J \subseteq B} (-1)^{|J|} |X_J|$. If $|J| = k$ then $|X_J| = (n-k)^m$. The result follows. ☐

Mappings can be "composed". Given $f\colon A \mapsto B$ and $g\colon B \mapsto C$ we can define $gf\colon A \mapsto C$ by $gf(a) = g(f(a))$. If f and g are injective then so is gf, similarly for surjectivity. If we also have $h\colon C \mapsto D$, then associativity of composition is easily verified : $(hg)f \equiv h(gf)$.

3.3 Permutations

A permutation of A is a bijection $f\colon A \mapsto A$. One notation is

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 4 & 2 & 8 & 7 & 6 & 5 \end{pmatrix}.$$

The set of permutations of A is a group under composition, the symmetric group $\operatorname{sym} A$. If $|A| = n$ then $\operatorname{sym} A$ is also denoted S_n and $|\operatorname{sym} A| = n!$. S_n is not abelian — you can come up with a counterexample yourself. We can also think of permutations as directed graphs, in which case the following becomes clear.

Proposition. *Any permutation is the product of disjoint cycles.*

We have a new notation for permutations, cycle notation.[1] For our function f above, we write

$$f = (1)(2\ 3\ 4)(5\ 8)(6\ 7) = (2\ 3\ 4)(5\ 8)(6\ 7).$$

3.3.1 Stirling numbers of the first kind

Definition. *$s(n, k)$ is the number of permutations of $\{1, \ldots, n\}$ with precisely k cycles (including fixed points).*

For instance $s(n, n) = 1$, $s(n, n-1) = \binom{n}{2}$, $s(n, 1) = (n-1)!$, $s(n, 0) = s(0, k) = 0$ for all $k, n \in \mathbb{N}$ but $s(0, 0) = 1$.

Lemma 3.1.
$$s(n, k) = s(n-1, k-1) + (n-1)s(n-1, k)$$

Proof. Either the point n is in a cycle on its own ($s(n-1, k-1)$ such) or it is not. In this case, n can be inserted into any of $n-1$ places in any of the $s(n-1, k)$ permutations of $\{1, \ldots, n-1\}$. □

We can use this recurrence to prove this proposition. (Proof left as exercise.)

Proposition.
$$x^{\overline{n}} = \sum_k s(n, k)x^k$$

3.3.2 Transpositions and shuffles

A transposition is a permutation which swaps two points and fixes the rest.

Theorem 3.2. *Every permutation is the product of transpositions.*

Proof. Since every permutations is the product of cycles we only need to check for cycles. This is easy: $(i_1\ i_2\ \ldots\ i_k) = (i_1\ i_2)(i_2\ i_3)\ldots(i_{k-1}\ i_k)$. □

Theorem 3.3. *For a given permutation π, the number of transpositions used to write π as their product is either always even or always odd.*

We write $\operatorname{sign}\pi = \begin{cases} +1 & \text{if always even} \\ -1 & \text{if always odd} \end{cases}$. We say that π is an $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$ permutation.

Let $c(\pi)$ be the number of cycles in the disjoint cycle representation of π (including fixed points).

Lemma 3.2. *If $\sigma = (a\ b)$ is a transposition that $c(\pi\sigma) = c(\pi) \pm 1$.*

Proof. If a and b are in the same cycle of π then $\pi\sigma$ has two cycles, so $c(\pi\sigma) = c(\pi)+1$. If a and b are in different cycles then they contract them together and $c(\pi\sigma) = c(\pi) - 1$. □

Proof of theorem 3.3. Assume $\pi = \sigma_1 \ldots \sigma_k \iota = \tau_1 \ldots \tau_l \iota$. Then $c(\pi) = c(\iota) + k \equiv c(\iota) + l \pmod 2$. Hence $k \equiv l \pmod 2$ as required. □

[1] See the Algebra and Geometry course for more details.

We note that $\operatorname{sign}\pi = (-1)^{n-c(\pi)}$, thus $\operatorname{sign}(\pi_1\pi_2) = \operatorname{sign}\pi_1 \operatorname{sign}\pi_2$ and thus sign is a homomorphism from S_n to $\{\pm 1\}$.

A k-cycle is an even permutation iff k is odd. A permutation is an $\begin{matrix}\text{even}\\\text{odd}\end{matrix}$ permutation iff the number of even length cycles in the disjoint cycle representation is $\begin{matrix}\text{even}\\\text{odd}\end{matrix}$.

3.3.3 Order of a permutation

If π is a permutation then the order of π is the least natural number n such that $\pi^n = \iota$. The order of the permutation π is the lcm of the lengths of the cycles in the disjoint cycle decomposition of π.

In card shuffling we need to maximise the order of the relevant permutation π. One can show (see) that for π of maximal length we can take all the cycles in the disjoint cycle representation to have prime power length. For instance with 30 cards we can get a $\pi \in S_{30}$ with an order of 4620 (cycle type 3 4 5 7 11).

3.3.4 Conjugacy classes in S_n

Two permutations $\alpha, \beta \in S_n$ are conjugate iff $\exists \pi \in S_n$ such that $\alpha = \pi\beta\pi^{-1}$.

Theorem 3.4. *Two permutations are conjugate iff they have the same cycle type.*

This theorem is proved in the Algebra and Geometry course. We note the corollary that the number of conjugacy classes in S_n equals the number of partitions of n.

3.3.5 Determinants of an $n \times n$ matrix

In the Linear Maths course you will prove that if $A = (a_{ij})$ is an $n \times n$ matrix then

$$\det A = \sum_{\pi \in S_n} \operatorname{sign}\pi \prod_{j=1}^{n} a_{j\,\pi(j)}.$$

3.4 Binary Relations

A binary relation on a set S is a property that any pair of elements of S may or may not have. More precisely:

Write $S \times S$, the Cartesian square of S for the set of pairs of elements of S, $S \times S = \{(a, b) : a, b \in S\}$. A binary relation $\mathcal{R}$ on S is a subset of $S \times S$. We write $a \mathrel{\mathcal{R}} b$ iff $(a, b) \in \mathcal{R}$. We can think of $\mathcal{R}$ as a directed graph with an edge from a to b iff $a \mathrel{\mathcal{R}} b$.

A relation $\mathcal{R}$ is:

- reflexive iff $a \mathrel{\mathcal{R}} a \;\forall a \in S$,
- symmetric iff $a \mathrel{\mathcal{R}} b \Rightarrow b \mathrel{\mathcal{R}} a \;\forall a, b \in S$,
- transitive iff $a \mathrel{\mathcal{R}} b, b \mathrel{\mathcal{R}} c \Rightarrow a \mathrel{\mathcal{R}} c \;\forall a, b, c \in S$,
- antisymmetric iff $a \mathrel{\mathcal{R}} b, b \mathrel{\mathcal{R}} a \Rightarrow a = b \;\forall a, b \in S$.

The relation $\mathcal{R}$ on S is an equivalence relation if it is reflexive, symmetric and transitive. These are "nice" properties designed to make $\mathcal{R}$ behave something like $=$.

Definition. *If $\mathcal{R}$ is a relation on S, then*

$$[a]_{\mathcal{R}} = [a] = \{b \in S : a \mathrel{\mathcal{R}} b\}.$$

If $\mathcal{R}$ is an equivalence then these are the equivalence classes.

Theorem 3.5. *If $\mathcal{R}$ is an equivalence relation then the equivalence classes form a partition of S.*

Proof. If $a \in S$ then $a \in [a]$, so the classes cover all of S. If $[a] \cap [b] \neq \emptyset$ then $\exists c \in [a] \cap [b]$. Now $a \mathrel{\mathcal{R}} c$ and $b \mathrel{\mathcal{R}} c \Rightarrow c \mathrel{\mathcal{R}} b$. Thus $a \mathrel{\mathcal{R}} b$ and $b \in [a]$. If $d \in [b]$ then $b \mathrel{\mathcal{R}} d$ so $a \mathrel{\mathcal{R}} d$ and thus $[b] \subseteq [a]$. We can similarly show that $[a] \subseteq [b]$ and thus $[a] = [b]$. $\square$

The converse of this is true: if we have a partition of S we can define an equivalence relation on S by $a \mathrel{\mathcal{R}} b$ iff a and b are in the same part.

An application of this is the proof of Lagrange's Theorem. The idea is to show that being in the same (left/right) coset is an equivalence relation.

Given an equivalence class on S the quotient set is $S/\mathcal{R}$, the set of all equivalence classes. For instance if $S = \mathbb{R}$ and $a \mathrel{\mathcal{R}} b$ iff $a - b \in \mathbb{Z}$ then $S/\mathcal{R}$ is (topologically) a circle. If $S = \mathbb{R}^2$ and $(a_1, b_1) \mathrel{\mathcal{R}} (a_2, b_2)$ iff $a_1 - a_2 \in \mathbb{Z}$ and $b_1 - b_2 \in \mathbb{Z}$ the quotient set is a torus.

Returning to a general relation $\mathcal{R}$, for each $k \in \mathbb{N}$ we define

$$\mathcal{R}^{(k)} = \{(a, b) : \text{there is a path of length at } k \text{ from } a \text{ to } b\}.$$

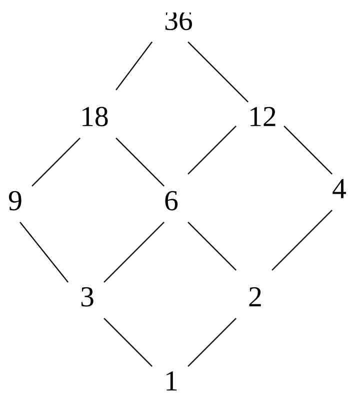
$\mathcal{R}^{(1)} = \mathcal{R}$ and $\mathcal{R}^{(\infty)} = t(R)$, the transitive closure of $\mathcal{R}$. $\mathcal{R}^{(\infty)}$ is defined as $\bigcup_{i \geq 1} \mathcal{R}^{(i)}$.

3.5 Posets

$\mathcal{R}$ is a (partial) order on S if it is reflexive, anti-symmetric and transitive. The set S is a poset (partially ordered set) if there is an order $\mathcal{R}$ on S.

We generally write $a \leq b$ iff $(a, b) \in \mathcal{R}$, and $a < b$ iff $a \leq b$ and $a \neq b$.

Consider D_n, the set of divisors of n. D_n is partially ordered by division, $a \leq b$ if $a \mid b$. We have the Hasse diagram, in this case for D_{36}:



A descending chain is a sequence $a_1 > a_2 > a_3 > \dots$. An antichain is a subset of S with no two elements directly comparable, for instance $\{4, 6, 9\}$ in D_{36}.

Proposition. *If S is a poset with no chains of length $> n$ then S can be covered by at most n antichains.*

Proof. Induction on n. Take $n > 1$ and let M be the set of all maximal elements in S. Now $S \setminus M$ has no chains of length $> n - 1$ and M is an antichain. □

3.5.1 Products of posets

Suppose A and B are posets. Then $A \times B$ has various orders; two of them being

- product order: $(a_1, b_1) \leq (a_2, b_2)$ iff $a_1 \leq a_2$ and $b_1 \leq b_2$,
- lexicographic order: $(a_1, b_1) \leq (a_2, b_2)$ if either $a_1 \leq a_2$ or if $a_1 = a_2$ then $b_1 \leq b_2$.

Exercise: check that these are orders.

Note that there are no infinite descending chains in $\mathbb{N} \times \mathbb{N}$ under lexicographic order. Such posets are said to be well ordered. The principle of induction follows from well-ordering as discussed earlier.

3.5.2 Eulerian Digraphs

A digraph is Eulerian if there is a closed path covering all the edges. A necessary condition is: the graph is connected and even (each vertex has an equal number of "in" and "out" edges). This is in fact sufficient.

Proposition. *The set of such digraphs is well-ordered under containment.*

Proof. Assume proposition is false and let G be a minimal counterexample. Let T be a non-trivial closed path in G, for instance the longest closed path. Now T must be even, so $G \setminus T$ is even. Hence each connected component of $G \setminus T$ is Eulerian as G is minimal. But then G is Eulerian: you can walk along T and include all edges of connected components of $G \setminus T$ when encountered — giving a contradiction. Hence there are no minimal counterexamples. □

3.6 Countability

Definition. *A set S is countable if either $|S| < \infty$ or $\exists$ a bijection $f : S \mapsto \mathbb{N}$.*

The countable sets can be equivalently thought of as those that can be listed on a line.

Lemma 3.3. *Any subset $S \subset \mathbb{N}$ is countable.*

Proof. For: map the smallest element of S to 1, the next smallest to 2 and so on. □

Lemma 3.4. *A set S is countable iff $\exists$ an injection $f : S \mapsto \mathbb{N}$.*

Proof. This is clear for finite S. Hence assume S is infinite. If $f : S \mapsto \mathbb{N}$ is an injection then $f(S)$ is an infinite subset of $\mathbb{N}$. Hence $\exists$ a bijection $g : f(S) \mapsto \mathbb{N}$. Thus $gf : S \mapsto \mathbb{N}$ is a bijection. □

An obvious result is that if S' is countable and $\exists$ an injection $f\colon S \mapsto S'$ then S is countable.

Proposition. *$\mathbb{Z}$ is countable.*

Proof. Consider $f\colon \mathbb{Z} \mapsto \mathbb{N}$,

$$f\colon x \mapsto \begin{cases} 2x+1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0. \end{cases}$$

This is clearly a bijection. □

Proposition. *$\mathbb{N}^k$ is countable for $k \in \mathbb{N}$.*

Proof. The map $(i_1, \dots, i_k) \mapsto 2^{i_1}3^{i_2}\dots p_k^{i_k}$ (p_j is the j^{th} prime) is an injection by uniqueness of prime factorisation. □

Lemma 3.5. *If $A_1, \dots, A_k$ are countable with $k \in \mathbb{N}$, then so is $A_1 \times \dots \times A_k$.*

Proof. Since A_i is countable there exists an injection $f_i\colon A_i \mapsto \mathbb{N}$. Hence the function $g\colon A_1, \dots, A_k \mapsto \mathbb{N}^k$ defined by $g(a_1, \dots, a_k) = (f_1(a_1), \dots, f_k(a_k))$ is an injection. □

Proposition. *$\mathbb{Q}$ is countable.*

Proof. Define $f\colon \mathbb{Q} \mapsto \mathbb{N}$ by

$$f\colon \frac{a}{b} \mapsto 2^{|a|}3^b5^{1+\operatorname{sign} a},$$

where $(a, b) = 1$ and $b > 0$. □

Theorem 3.6. *A countable union of countable sets is countable. That is, if I is a countable indexing set and A_i is countable $\forall i \in I$ then $\bigcup_{i \in I} A_i$ is countable.*

Proof. Identify first I with the subset $f(I) \subseteq \mathbb{N}$. Define $F\colon A \mapsto \mathbb{N}$ by $a \mapsto 2^n3^m$ where n is the smallest index i with $a \in A_i$, and $m = f_n(a)$. This is well-defined and injective (stop to think about it for a bit). □

Theorem 3.7. *The set of all algebraic numbers is countable.*

Proof. Let P_n be the set of all polynomials of degree at most n with integral coefficients. Then the map $c_nx^n + \dots + c_1x + c_0 \mapsto (c_n, \dots, c_1, c_0)$ is an injection from P_n to $\mathbb{Z}^{n+1}$. Hence each P_n is countable. It follows that the set of all polynomials with integral coefficients is countable. Each polynomial has finitely many roots, so the set of algebraic numbers is countable. □

Theorem 3.8 (Cantor's diagonal argument)**.** *$\mathbb{R}$ is uncountable.*

Proof. Assume $\mathbb{R}$ is countable, then the elements can be listed as

$$\begin{aligned} r_1 &= n_1.d_{11}d_{12}d_{13}\dots \\ r_2 &= n_2.d_{21}d_{22}d_{13}\dots \\ r_3 &= n_3.d_{31}d_{32}d_{33}\dots \end{aligned}$$

(in decimal notation). Now define the real $r = 0.d_1d_2d_3\dots$ by $d_i = 0$ if $d_{ii} \neq 0$ and $d_i = 1$ if $d_{ii} = 0$. This is real, but it differs from r_i in the i^{th} decimal place. So the list is incomplete and the reals are uncountable. □

Exercise: use a similiar proof to show that $P(\mathbb{N})$ is uncountable.

Theorem 3.9. *The set of all transcendental numbers is uncountable. (And therefore at least non-empty!)*

Proof. Let A be the set of algebraic numbers and T the set of transcendentals. Then $\mathbb{R} = A \cup T$, so if T was countable then so would $\mathbb{R}$ be. Thus T is uncountable. □

3.7 Bigger sets

The material from now on is starred.

Two sets S and T have the same *cardinality* ($|S| = |T|$) if there is a bijection between S and T. One can show (the Schröder-Bernstein theorem) that if there is an injection from S to T and an injection from T to S then there is a bijection between S and T.

For any set S, there is an injection from S to $P(S)$, simply $x \mapsto \{x\}$. However there is never a surjection $S \mapsto P(S)$, so $|S| < |P(S)|$, and so

$$|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < \dots$$

for some sensible meaning of $<$.

Theorem 3.10. *There is no surjection $S \mapsto P(S)$.*

Proof. Let $f\colon S \mapsto P(S)$ be a surjection and consider $X \in P(S)$ defined by $\{x \in S : x \notin f(x)\}$. Now $\exists x' \in S$ such that $f(x') = X$. If $x' \in X$ then $x' \notin f(x')$ but $f(x') = X$ — a contradiction. But if $x' \notin X$ then $x' \notin f(x')$ and $x' \in X$ — giving a contradiction either way. □

If there is an $\begin{array}{c}\text{injection}\\ \text{surjection}\end{array}$ $f\colon A \mapsto B$ then there exists a $\begin{array}{c}\text{surjection}\\ \text{injection}\end{array}$ $g\colon B \mapsto A$.

Moreover we can ensure that $\begin{array}{l} g \circ f = \iota_A \\ f \circ g = \iota_B \end{array}$

References

- Hardy & Wright, *An Introduction to the Theory of Numbers*, Fifth ed., OUP, 1988.
  This book is relevant to quite a bit of the course, and I quite enjoyed (parts of!) it.
- H. Davenport, *The Higher Arithmetic*, Sixth ed., CUP, 1992.
  A *very* good book for this course. It's also worth a read just for interest's sake.

I've also heard good things about Biggs' book, but haven't read it.